Matching with Frictions, and the Law of Demand

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Abstract

Models used in the literature on directed search are well known to exhibit a large number of equilibria. Using a standard model based on Burdett, Shi and Wright (Journal of Political Economy, 2001), this note shows that all trigger equilibria, that is, all equilibria in which buyers play pure strategies on the equilibrium path and symmetric mixed strategies off the equilibrium path, violate the law of demand. This provides a way of justifying the competitive search equilibrium, in which buyers play symmetric mixed strategies both on and off the equilibrium path, which has been the focus of the literature.

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1 Introduction

Consider the standard model in the directed search literature, where sellers first simultaneously post prices and where buyers simultaneously choose a seller after having observed these prices. A seller's payoff is his price if he attracts at least one buyer and 0 otherwise. A buyer's expected payoff at any given seller is the difference between her valuation and the seller's price, divided by the total number of buyers attracted by that seller. As is well known, this model has many equilibria that differ with respect to payoffs and efficiency (see e.g. Burdett, Shi, and Wright, 2001, BSW hereafter).

In particular, assuming equal numbers of homogenous buyers and homogenous sellers, there is a continuum of equilibria that are Pareto efficient and Pareto superior to the competitive search equilibrium and that are robust to small trembles in the sense used by BSW. In these equilibria, every buyer visits one specific seller with probability 1 on the equilibrium path. Deviations from a 'prescribed' equilibrium price trigger a symmetric mixed strategy equilibrium in the buyers subgame. Because the mixed strategy equilibrium involves coordination failures with positive probability, deviations by the sellers are successfully deterred. Naturally, these equilibria may be referred to as *trigger equilibria*.

Using a standard model with the same number of buyers and sellers based on BSW, this note shows that all trigger equilibria violate the law of demand. That is, in trigger equilibria the individual demand functions for any given seller are not everywhere non-increasing in this seller's price and not everywhere non-decreasing in competitors' prices. The reason is that in the mixed strategy equilibrium of the buyers subgame that follows upon a deviation, every buyer plays a non-degenerately mixed strategy, including those buyers who on the equilibrium path buy from a seller with probability 1 whose price has not changed. To the extent that the law of demand is deemed strong enough to balance off game-theoretic considerations, this note thus provides a way to refine away trigger equilibria.

The BSW model and variations on that model have been used extensively in the literatures on consumer, labor, and money markets with search (see e.g. Shimer, 2005; Guerrieri, 2008; Julien, Kennes, and King, 2008; Menzio and Shi, 2010). To the best of my knowledge, no paper has previously tackled the problem of refining away trigger equilibria in directed search models.[1]

The remainder of this note is structured as follows. Section 2 presents the setup. Results are derived in Section 3, and Section 4 contains a brief discussion.

2 Setup

There are n risk neutral buyers and n risk neutral sellers. All buyers have quasilinear preferences and a valuation of 1 for a homogenous good of known quality. All sellers have a capacity to produce 1 unit at marginal costs that are normalized to 0. The game has two stages. In the first stage, all sellers j simultaneously post prices p_j, $j = 1, ..n$. Without loss of generality, assume that prices are restricted to be elements of $[0, 1]$, and denote by $\mathbf{p} \equiv (p_1, .., p_n) \in [0, 1]^n$ the collection of all prices, and by $\mathbf{p}_{-j}$ all prices other than seller j's. In the second stage, which will also be referred to as the buyer subgame, after observing $\mathbf{p}$ all buyers simultaneously choose a seller. Formally, a (possibly degenerately) mixed strategy θ^i for buyer i is a mapping of prices into a probability distribution over sellers: $\theta^i : [0, 1]^n \to \Delta$, where Δ is the $n - 1$ dimensional

[1] Galenianos and Kircher (forthcoming)'s study is somewhat similar in spirit to the present one. However, the objectives are rather complementary: Galenianos and Kircher's purpose is to construct an equilibrium whereas mine is get rid off (many) equilibria.

simplex. Denote the probability that buyer i visits seller j given prices $\mathbf{p}$ by $\theta^i_j(\mathbf{p})$. For every buyer i we have $\sum_{j=1}^n \theta^i_j(\mathbf{p}) = 1$.[2] If k other buyers show up at the same seller j, each buyer at seller j is served with probability $\frac{1}{1+k}$, so that the payoff of each of these buyers is $\frac{1}{1+k}(1-p_j)$. Seller j's payoff is p_j if at least one buyer shows up and 0 otherwise. A buyer strategy profile $\{\theta^i(\mathbf{p})\}_{i=1}^n$ is said to be symmetric if $\theta^i_j(\mathbf{p}) = \theta_j(\mathbf{p})$ for every buyer $i = 1,..,n$ and for every $\mathbf{p} \in [0,1]^n$. As in BSW, the focus here is on subgame perfect equilibria, in which all sellers play pure strategies.

Laws of Demand Buyer i's strategy $\theta^i(\mathbf{p})$ is said to satisfy the **law of demand** (LOD) if, for every j, $\theta^i_j(\mathbf{p})$ weakly decreases in p_j and weakly increases in p_k for every $k \neq j$. A buyer strategy profile $\{\theta^i(\mathbf{p})\}_{i=1}^n$ is accordingly said to satisfy LOD if LOD is satisfied by every buyer. Let $D_j(\mathbf{p}) \equiv \sum_{i=1}^n \theta^i_j(\mathbf{p})$ be the expected number of buyers who visit seller j. A strategy profile is said to satisfy the **aggregate law of demand** (ALOD) if for every seller j, $D_j(\mathbf{p})$ weakly decreases in p_j and weakly increases in p_k, $k \neq j$.

Competitive Search Equilibrium Under the assumption that there are m sellers and n buyers BSW derive the competitive search equilibrium, which is the unique equilibrium where all m sellers post the price

$$p(m,n) = \frac{1 - \left(1 + \frac{n}{m-1}\right)\left(\frac{m-1}{m}\right)^n}{\left(1 + \frac{n}{m(m-1)}\right)\left(\frac{m-1}{m}\right)^n}, \tag{1}$$

and where every buyer visits each seller with probability $1/m$ on the equilibrium path. For notational ease, let $p^* \equiv p(n,n)$.

3 Results

3.1 Buyer Subgame

Consider first the buyer subgame. Notice that visiting seller k is strictly dominated by visiting seller j if $(1-p_k) < \frac{1}{n}(1-p_j)$. So choosing seller k is an undominated strategy if and only if $p_k < \frac{n-1}{n} + \frac{1}{n}\min\{\mathbf{p}_{-k}\}$. Let

$$P^{UD} \equiv \left\{\mathbf{p} | p_j < \frac{n-1}{n} + \frac{1}{n}\min\{\mathbf{p}_{-j}\} \quad \forall j = 1,..,n\right\} \tag{2}$$

be the set of undominated prices. For $n = 2$, this is the same as in the $2x2$ example of BSW since $P^{UD} \equiv \left\{\mathbf{p}|p_j < \frac{1}{2} + \frac{1}{2}p_{-j} \quad \forall j = 1,2\right\}$ where p_{-j} is the price of the seller other than j.

Assuming all other buyers are playing a symmetric strategy and visit seller j with probability $\theta_j(\mathbf{p})$, the expected payoff of visiting seller j is

$$U_j(p_j, \theta_j) \equiv \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{1+k} \theta_j^k (1-\theta_j)^{n-1-k}(1-p_j). \tag{3}$$

[2] Notice that this formulation rules out the possibility of abstention. This is almost without loss of generality as not buying is only a (weakly) beneficial option if all sellers set $p_j = 1$.

Observe that $U_j(p_j, 0) = (1 - p_j)$ and $U_j(p_j, 1) = \frac{1}{n}(1 - p_j)$. Moreover,

$$\frac{\partial U_j(p_j,\theta_j)}{\partial \theta_j} = -(1-p_j)\frac{(1-\theta_j)^{n-1}}{n\theta_j^2}\left\{(1-\theta_j)\left[\left(\frac{1}{1-\theta_j}\right)^n - 1\right] - n\theta_j\right\} < 0, \tag{4}$$

where the inequality is easiest established by noticing that $(1-\theta_j)\left[\left(\frac{1}{1-\theta_j}\right)^n - 1\right] - n\theta_j$ is increasing in n and equal to 0 at $n = 1$. Observe also that $U_j(p_j, \theta_j)$ is continuous in p_j.

Of particular relevance is the symmetric mixed strategy equilibrium of the buyer subgame, that is, the equilibrium where all buyers visit seller j with the same probability $\theta_j^*(\mathbf{p})$ for all $j = 1, .., n$.

Lemma 1 *For any $\mathbf{p} \in P^{UD}$ a symmetric mixed strategy equilibrium $\theta^* \equiv (\theta_1^*(\mathbf{p}), .., \theta_n^*(\mathbf{p}))$ exists and satisfies the following properties:*

1. *$\theta_j^*(\mathbf{p}) < 1$ for all $j = 1, .., n$.*
2. *$\theta_j^*(\mathbf{p})$ is continuous in $\mathbf{p}$, decreasing in p_j and increasing in p_k for all $k \neq j$ and all $j = 1, .., n$.*
3. *$\theta_j^*(\mathbf{p}) = \frac{1}{n}$ for all $j = 1, .., n$ is equivalent to $p_j = p$ for all $j = 1, .., n$.*
4. *The expected number of buyers at seller j is $D_j(\mathbf{p}) = n\theta_j^*(\mathbf{p})$.*

The inequality in 1. implies that this an equilibrium where all buyers are playing non-degenerate mixed strategies. Existence of such an equilibrium, though readily established, is neither implied by the analysis in BSW because they focus on equilibria where all prices are the same nor by the theorems on the existence of Nash equilibrium because here buyers' strategies are restricted to be symmetric.

Proof of Lemma 1: The existence proof is a straightforward application of Brouwer's fixed point theorem. Let $\theta = (\theta_1, .., \theta_n)$ be a symmetric mixed strategy. Notice that $\sum_{j=1}^n \theta_j = 1$, so $\theta \in \Delta$, where Δ is the $(n-1)$-dimensional simplex. So as to invoke Brouwer's theorem, I am now going to define a continuous function $f : \Delta \to \Delta$. Let $f(\theta) = (f_1(\theta), .., f_n(\theta))$ be the vector valued function whose i-th element is

$$f_i(\theta) = \frac{\theta_i + \max\{0, u_i(\theta) - \max_{j\neq i} u_j(\theta)\}}{1 + \sum_{h=1}^n \max\{0, u_h(\theta) - \max_{j\neq h} u_j(\theta)\}}, \tag{5}$$

where $u_i(\theta)$ is short-hand for $u_i(\theta) \equiv (1-p_j)\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{1}{1+k}\theta_j^k(1-\theta_j)^{n-1-k}$. Notice that $f_i(\theta) \geq 0$ and $\sum_{i=1}^n f_i(\theta) = 1$. Thus, f maps elements from Δ into Δ. Moreover, f is continuous in θ because the u_i's are continuous in θ. Thus, a fixed point $\theta^*(\mathbf{p}) = f(\theta^*(\mathbf{p}))$ exists. Such a fixed point is an equilibrium because for every i and j such that $\theta_i^*(\mathbf{p}) > 0$ and $\theta_j^*(\mathbf{p}) > 0$, $u_i(\theta^*(\mathbf{p})) = u_j(\theta^*(\mathbf{p}))$ holds while for every h with $\theta_h^*(\mathbf{p}) = 0$, $u_h(\theta^*(\mathbf{p})) \leq u_j(\theta^*(\mathbf{p}))$ holds. Thus, a symmetric mixed strategy equilibrium exists.

To see that 1. holds, suppose to the contrary that $\theta_i^*(\mathbf{p}) = 1$. But now $\mathbf{p} \in P^{UD}$ implies that $u_j(\theta^*(\mathbf{p})) > u_i(\theta^*(\mathbf{p}))$ for all $j \neq i$, which is a contraction to $\theta^*(\mathbf{p})$ being a fixed point.

As for 2., continuity of $\theta^*(\mathbf{p})$ is inherited from the continuity of the u_i's in p_i. For any i such that $\theta_i^*(\mathbf{p}) > 0$ before and after the price change, an increase in p_i must result in a decrease of $\theta_i^*(\mathbf{p})$ so as to maintain the indifference condition $u_i = u_k$ for k such that $\theta_k(\mathbf{p}) > 0$. But a decrease in θ_i^* implies that all the other sellers k must now be visited with (weakly) higher probability, that is, $\theta_k^*(\mathbf{p})$ increases in p_j for $j \neq k$.[3]

That $p_j = p$ for all j implies $\theta_j = 1/n$ follows from the monotonicity property of u_i in θ_i: If, say, $\theta_i^* < \theta_k^*$, $u_i > u_k$ would hold, implying $f_i(\theta^*) > \theta_i^*$, which is a contradiction. Conversely, that $\theta_j^*(\mathbf{p}) = \frac{1}{n}$ for all $j = 1,..,n$ implies $p_j = p$ for all j follows from the monotonicity properties of $\theta_j^*(\mathbf{p})$ in p_j and p_k stated in 2.

The property $D_j(\mathbf{p}) = n\theta_j(\mathbf{p})$ follows from the independence property of Nash equilibrium strategies. ■

3.2 Trigger Equilibria

If buyers play the symmetric mixed strategy equilibrium when prices are $\mathbf{p}$, seller j's expected payoff is $\pi_j(p_j, \mathbf{p}_{-j}) \equiv p_j[1 - (1 - \theta_j^*(\mathbf{p}))^n]$ since he sells with probability 1 whenever at least one buyer visits him and with probability 0 otherwise. Let $\overline{\pi}_j(\mathbf{p}_{-j}) \equiv \max_{p_j \in [0,1]} \pi_j(p_j, \mathbf{p}_{-j})$.[4]

I call an equilibrium a trigger equilibrium if, on the equilibrium path, all buyers play the pure strategies of visiting a distinct seller with probability 1 and where off the equilibrium path all buyers play the symmetric mixed strategy equilibrium. To simplify the notation and without loss of generality, for $\mathbf{p}$ on the equilibrium path I focus on the pure strategies $\theta_j^j(\mathbf{p}) = 1$ for all $j = 1,..,n$. That is, every buyer visits the seller with the same label with probability 1.

Lemma 2 *Any* $\mathbf{p}$ *satisfying (i)* $p_j \geq \overline{\pi}_j(\mathbf{p}_{-j})$ *and (ii)* $p_j \leq \frac{1}{2}(1 + p_k)$ *for all* $j = 1,..,n$ *and all* $k \neq j$ *is the outcome of a trigger equilibrium, and no other* $\mathbf{p}$ *is.*

Proof of Lemma 2: Restriction (i) makes sure that no seller has an incentive to deviate from the prescribed equilibrium price by triggering the mixed strategy equilibrium in the buyer subgame. Restriction (ii) is an incentive constraint for buyers and makes sure that, given equilibrium prices $\mathbf{p}$, buyer j has no incentive to deviate and visit seller k who, on the equilibrium path, is chosen by buyer k with probability 1 and by any other buyer with probability 0.■

Denote the set of prices that can be supported as outcomes of trigger equilibria by

$$P^T \equiv \{\mathbf{p} | p_j \geq \overline{\pi}_j(\mathbf{p}_{-j}) \cap p_j \leq \frac{1}{2}(1 + p_k) \quad \forall j = 1,..,n, k \neq j\}. \tag{6}$$

To see that P^T is non-empty, it suffices to notice $p_j = p^*$ for all j can be supported as the outcome of a trigger equilibrium because by construction of p^*, $\overline{\pi}_j = p^*\left[1 - \left(\frac{n-1}{n}\right)^n\right] < p^*$. Figure 1 illustrates the set of trigger equilibria P^T for $n = 2$ in the shaded area. The set P^{UD} corresponds to prices (p_1, p_2) between the two straight lines. The directed search equilibrium price p^* is 1/2.

Establishing that trigger equilibria exist that are robust to small trembles is not too hard but requires some additional structure. Let p_j^* be the price seller j is supposed to set on the equilibrium path of a trigger equilibrium and assume that the prices of all sellers are subject to small trembles in the sense that the realized price p_j is uniformly distributed on $[p_j^* - \varepsilon, p_j^* + \varepsilon]$ with $\varepsilon > 0$ small for all $j = 1,..,n$. Since trigger equilibria rest on the successful deterrence of (upward) deviations by sellers by buyers' reverting to the mixed strategy equilibrium, I assume

[3]The only way θ_k^* could decrease in p_j would be that now sellers are visited with positive probability who previously were not. Though the latter is possible, it is still not possible that θ_k^* decrease in p_j because that would mean that u_j increases, contradicting that it was previously optimal not to visit sellers who are now visited with positive probability and who thus generate an expected payoff of u_j to buyers.

[4]That such a maximum exists follows from the continuity of θ_j^* and from the fact if p_j becomes too large, visiting seller j will be a dominated strategy.

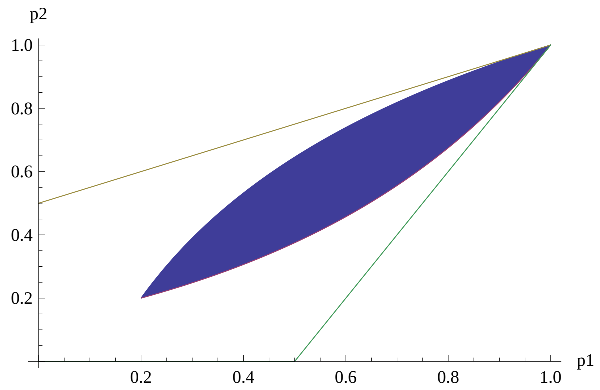

Figure 1: The Sets P^T (shaded) and P^{UD} for $n = 2$.

also that buyers can distinguish between deviations that are due to trembles and deliberate (or strategic) deviations by a seller.[5] As long as ε is small, it is still a best response for buyer j to visit seller j with probability 1 for 'most' trigger equilibria because the price differentials the trembles induce are second order to the effect of visiting a different seller and thereby doubling the number of buyers there.[6] This is in contrast to the equilibrium where all sellers set prices of 1 because then visiting a seller with a price of less than 1 strictly dominates visiting a seller with a price of 1.

The main result of this note now follows from Lemma 1:

Proposition 1 *No trigger equilibrium satisfies LOD. A trigger equilibrium satisfies ALOD if and only if it satisfies* $p_j = p$ *for all* j*.*

Proof of Proposition 1: That no trigger equilibrium satisfies LOD follows immediately from property 1. of Lemma 1 and the fact that on the equilibrium path each buyer visits a distinct seller with probability 1: Following an upward deviation by some seller k, every buyer $j \neq k$ now visits seller j with probability $\theta_j(\mathbf{p}) < 1$, thereby violating LOD.

On the equilibrium path of a trigger equilibrium, the (expected) number of buyers every seller j attracts is 1. In the mixed strategy equilibrium off the equilibrium path, it is $n\theta_j(\mathbf{p})$ (Lemma 1, 4.). So for $p_j = p$ for all j, following an arbitrarily small upward deviation by some seller, every non-deviating seller's expected number of buyers is slightly larger than 1 and approaches 1 from above as the upward deviation by the deviating seller goes to 0 because $\theta_j^*(\mathbf{p}) = 1/n$ for all j (Lemma 1, 3.). Since buyers' strategies in the mixed strategy equilibrium satisfy LOD globally (see again Lemma 1, 3.), this proves that trigger equilibria with prices $p_j = p$ for all j satisfy ALOD. To see that no other trigger equilibria satisfy ALOD, consider a trigger equilibrium where seller j has the highest price (some other sellers may have the same price) and consider a small upward deviation by seller k whose price satisfies $p_k < p_j$. Then,

[5]Such an assumption is not necessary when exposing the equilibrium where all sellers post prices of 1 and where, on the equilibrium path, every buyer visits a different seller with probability 1 because sellers already post the highest possible price. Alternatively, one could confine attention to trembles p_j that are uniformly distributed over $[p_j^* - \varepsilon, p_j^*]$.

[6]To make this statement more precise, holding fixed the equilibrium strategies of all other buyers, buyer i still optimally visits seller i rather than seller j for any realizations of p_j and p_i if and only if $(1 - p_i^* - \varepsilon) \geq (1 - p_j^* + \varepsilon)/2 \Leftrightarrow p_i^* \leq (1 + p_j^* - \varepsilon)/2$. Since this must hold for all i and j, in the limit as ε goes to 0 this reduces to the second condition in the definition of P^T. A symmetric mixed strategy equilibrium in the buyer subgame existing for any realized prices, the first condition in P^T can straightforwardly be extended to allow for trembles. For ε sufficiently small, $p^* \in P^T$ holds just as without trembles.

$\theta_j^*(\mathbf{p}) < 1/n$ will hold by Lemma 1, 2. Thus, seller j's expected demand will decrease following a unilateral price increase by seller k, which violates ALOD. ■

4 Discussion

Within a balanced market where the number of buyers equals the number of sellers this note shows that no equilibrium that rests on the buyers' off the equilibrium path threat to revert to a symmetric mixed strategy equilibrium in the buyers subgame satisfies the law of demand. Disregarding in addition equilibria that are not robust to small trembles, this leaves the competitive search equilibrium as the unique refined equilibrium, which is the equilibrium much of Burdett, Shi, and Wright (2001) and the subsequent literature has focused on.

In a perfectly competitive or Walrasian world, consumers take prices as given and choose their optimal consumption bundles without having to take into account what other consumers are likely to choose, and their demand functions satisfy the law of demand if goods are substitutes and if there are no income effects. A natural way of interpreting the BSW model, and the variations based on it, is that it preserves consumers' price taking behavior – after all, prices are given by the time buyers make their choices – without assuming that the problem of matching buyers and sellers is automatically solved. Since goods are substitutes and income effects are absent in the BSW model, imposing the law of demand on buyers' strategies seems like a natural restriction in the spirit of this model.

Though a complete analysis of unbalanced markets is beyond the scope of this note, it seems clear that nothing in the basic argument hinges on the market being balanced in the following way. Assume that there are n buyers and m sellers with $m \leq n$. As shown by BSW, this game has a competitive search equilibrium. In this equilibrium all sellers set the price $p(m, n)$ given in (1), and on the equilibrium path every buyer visits any given seller with probability $1/m$. There also exist trigger equilibria in which all sellers post the price $p(n, m)$ and sell with probability 1 on the equilibrium path and where on the equilibrium path buyers match to sellers with probability 1 in such a way that the difference in the number of buyers across any two sellers is at most 1.[7] Off the equilibrium path, buyers play a symmetric mixed strategy equilibrium. Clearly, this trigger equilibrium is Pareto efficient whereas the competitive search equilibrium is not. And clearly, buyers' strategies in this trigger equilibrium do not satisfy the law of demand.

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[7]For example, if $m = 3$ and $n = 10$, two sellers get three buyers and one seller gets four. This matching makes sure that given the equilibrium prices and matching, no buyer has an incentive to deviate.

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