Dominant Strategy, Double Clock Auctions with Estimation-Based Tâtonnement *

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Abstract

We introduce a double clock auction mechanism for a homogenous good market with multi-dimensional private information and multi-unit traders that never runs a deficit, is ex post individually rational and makes sincere bidding a dominant strategy equilibrium. In our double clock auction, the market maker estimates demand and supply using information from the bids of traders that have dropped out and follows an estimation-based tâtonnement process to adjust the clock prices. The design is flexible and allows the market maker to target efficiency, profit maximization or an intermediate objective. Quantity constraints on traders may also be incorporated. Under mild regularity conditions convergence to the target goal, efficiency or the profit maximizing posted-price outcome, obtains as the market size grows.

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1 Introduction

As emphasized by Ausubel (2004), two fundamental prescriptions for practical auction design derived from the auction literature are that the prices paid by an agent ought to be as independent as possible from her own bids and that the auction should be structured in an open, dynamic, fashion, so as to convey clear price information to bidders and to preserve the privacy of the winners' valuations.[1] Under the latter property, market participants are protected from hold-up by the designer, because they do not reveal their willingness to pay on units they trade, and the designer is protected from the often substantial political and public risk of ex post regret – not knowing the agents' willingness to pay it is much more difficult to make the case that there was "money left on the table."[2] A third lesson from the auction literature is that often only minor changes to an auction designed to allocate resources efficiently are needed when the goal is revenue extraction, or profit maximization by the market maker (e.g., a carefully chosen reserve price, Myerson, 1981).

In an auction, one side of the market is passive; the seller plays no active role when a given quantity is being sold nor does the buyer in an auction to procure a fixed amount. In many markets, including those in the booming electronic market business, however, there is neither a fixed supply, nor a fixed demand. It is thus both natural and important to ask whether, when demand and supply are variable, market mechanisms can be designed which target either efficiency or profit, guarantee that agents cannot influence the price they pay and with price discovery occurring in an open fashion. In this paper we will provide positive answers to these questions, by proposing a market mechanism that satisfies all three requirements.

The study of price formation and market making with variable demand and supply has a long tradition and lies at the very heart of economics. The focus of the early literature was on efficient resource allocation. Walras (1874) proposed a procedure, called tâtonnement, in which buyers and sellers quote their demands and supplies at a given price to an auctioneer that increases the price if there is excess demand and decreases it if there is excess supply, with transactions only taking place when equilibrium is reached. One important problem with the Walrasian tâtonnement is that agents do not have an incentive to report truthfully their demand and supply schedules, as their reports affect the final price.

[1]Several authors have argued in favor of dynamic allocation mechanisms. See for example Ausubel (2004, 2006), Perry and Reny (2005), Bergemann and Morris (2007), Milgrom and Segal (2015) and Sun and Yang (2009, 2014). Perry and Reny (2005, p.568), for example, argue that "simultaneous auction formats tend to treat information as if it were costless to collect and costless to provide" while dynamic auctions economize on the information collected.

[2]The practice of mechanism design and historical experience with auctions offer plenty of examples. The 1990 spectrum license auction in New Zealand is one famous example of political risk due to ex post regret (see, for example, McMillan (1994) or Milgrom (2004). That static, sealed bids, mechanisms are prone to the bidders' hold-up problem was known by stamp collectors before the middle of the 20th century (Lucking-Reiley, 2000).

In his landmark paper, Vickrey (1961) showed that it is possible to elicit the true demands and supplies and implement the efficient allocation, using a generalization of his static auction. But because it must run a deficit and hence be financed by an outside source, Vickrey was skeptical about the practical relevance of the market mechanism he proposed, calling it "inordinately expensive" for the market maker. Vickrey did not see an easy way to modify it so as to avoid the deficit, preserve the truth telling property and achieve an approximately efficient allocation, noting (Vickrey, 1961, p.13-14, emphasis added):

> It is tempting to try to modify this scheme in various ways that would reduce or eliminate this cost of operation while still *preserving the tendency to optimum resource allocation*. However, it seems that all modifications that do diminish the cost of the scheme *either imply the use of some external information as to the true equilibrium price or reintroduce a direct incentive for misrepresentation* of the marginal-cost and marginal-value curves. To be sure, in some cases the impairment of optimum allocation would be small relative to the reduction in cost, but, unfortunately, the analysis of such variations is extremely difficult; ...

While many markets are designed to allocate resources efficiently, Internet trading platforms like eBay and Amazon or business-to-business platforms and other private markets are driven by profit maximization.[3] The ability to accommodate profit objectives is also relevant in some public markets, one example being the upcoming "incentive auction" in the United States, where the market maker - the Federal Communications Commissions - is required to raise strictly positive revenue to finance various public services.[4] With a profit maximizing market maker, eliciting information about true demands and supplies is also a challenge and preserving the privacy of the winners' valuations and costs is possibly even more important.

In this paper, we propose a novel, double clock, trading mechanism that induces price taking behavior by all buyers and sellers at all times and hence elicits revelation of the true demands and supplies, without running a deficit. We stress that our traders have multi-unit demands and supplies and multi-dimensional private information about their marginal values and costs. Our double clock auction may target either efficiency, in which case we view it as a solution to the challenges identified by Vickrey, or maximization of the market maker's profit. Under mild regularity conditions, we show that in the limit, as the number of traders grows, our double clock auction achieves its target and converges either to the efficient outcome or to the optimal mechanism of a profit maximizing market maker that knows the valuations and costs of all traders and posts a uniform price on each side of the market.

[3] Profit maximization is an ubiquitous assumption in the recent literature on two-sided markets; see, e.g., Caillaud and Jullien (2001, 2003), Rochet and Tirole (2002, 2006), Armstrong (2006), or Gomes (2014).

[4] See, e.g., Milgrom and Segal (2015). More generally, revenue considerations have been pivotal for market-based mechanisms in policy debates; see, e.g., Loertscher, Marx, and Wilkening (2015).

Our double clock auction is composed of two phases. In each phase, there is an ascending clock price for buyers and a descending clock price for sellers; traders simply indicate demand and supply at their clock price. During the first phase, called the discovery phase, no items are allocated; instead, the market maker estimates aggregate demand and supply and adjusts the movement of the clock prices. In estimating demand and supply, the designer only uses bid information from traders who have dropped out by reducing their demand or supply to zero. There are instances when only one of the two clock prices moves. In a tâtonnement fashion, this occurs when *estimated* excess demand at the current clock prices is not zero. If, for example, there is estimated excess supply, the designer computes the sellers' *target price* at which estimated supply equals estimated demand at the current buyers' clock price, keeps the buyers' clock price constant, and has the sellers' clock price decrease until it either reaches the target or a seller drops out. When estimated excess demand at the current clock prices is zero, both clock prices move until they reach their target prices or a trader drops out.[5] If the market maker targets efficiency, he selects as a target for both sides the price at which estimated excess demand is zero. When targeting profit, the market maker selects different target prices for buyers and sellers; these target prices equate estimated marginal cost and marginal revenue of the market maker, on top of balancing estimated demand and supply.

The discovery phase ends after finitely many steps, when the target price is reached on each side of the market. These final target prices become the *reserve prices* for the next phase of the double clock auction, the allocation phase. The market maker selects as the quantity to be traded in the allocation phase the minimum of the *true* demand and supply at the reserve prices, as revealed by traders at the end of the discovery phase. In the allocation phase, the market maker runs an Ausubel (2004) auction on each side of the market augmented by the reserve price determined in the discovery phase. Only on the long side of the market is the auction not trivial; on the short side, it ends immediately with all trades taking place at the reserve price.

Since no information about the activity of other bidders is released to any agent, in our double clock auction agents bid "sincerely" – it is a dominant strategy equilibrium for every agent to bid according to their true demand and supply function, that is, to stay active on a unit until the clock price reaches their marginal value or cost for that unit and to reduce activity by one unit at this point. The key to maintaining dominant strategy incentive compatibility is that the double clock auction only uses information revealed by agents who have exited the auction to estimate demand and supply and thereby determine reserve prices. This feature also explains the nature of the regularity conditions required for the equilibrium outcome to converge to optimality as the market becomes large: The information inferred from the behaviour of the agents who drop out early must be statistically informative about

[5] When a seller or a buyer drops out, the clocks temporarily stop and demand and supply are re-estimated.

market demand and supply of the remaining active agents.

An important feature of our double clock auction design is its flexibility, as it permits the market designer to pursue either efficiency, or profit maximization.[6] Our double clock auction can also accommodate quantity constraints on the total number of units subsets of buyers or sellers trade. Constraints like these can arise for a variety of reasons such as a desire to limit the market power of some agents within the mechanism or downstream, or because of technological constraints. For example, during the build-up for the "incentive auction" in the United States, the question whether the telecom companies Verizon and AT&T should be subject to a cap was debated (see, e.g., Marx, 2013).

Our paper is related to several strands of the literature. First, broadly speaking it belongs to the literature on dominant strategy mechanisms in the tradition of Vickrey (1961), Clarke (1971) and Groves (1973); indeed we show that the outcome of our double clock auction is equivalent to the outcome of a direct two-sided Vickrey-Clarke-Groves (VCG) mechanism with endogenous reserve prices. More specifically, the paper contributes to the literature on incentive compatible mechanisms based on clock auctions, initiated with Ausubel's (2004) analysis of the one-sided allocation problem.[7] Like Ausubel, Cramton and Milgrom (2006) with their clock-proxy auction, when targeting efficiency we view our double clock auction as a practical implementation of the fictitious "Walrasian auctioneer". Our setting, however, is quite different from Ausubel, Cramton and Milgrom (2006) – our double clock auction deals with the two-sided nature of the market making problem with buyers and sellers having multi-dimensional private information and we do not allow final, simultaneous, or proxy, bids.[8]

Second, our paper contributes to the literature on mechanism design with estimation initiated by Baliga and Vohra (2003) and Segal (2003), which has become particularly popular in the computer science literature; see Loertscher and Marx (2016) for additional references. Thus far, this strand of literature has adopted a Bayesian mechanism design framework with one-dimensional types and used estimation to attain Bayesian optimality in the limit. For setups with one-dimensional types, Loertscher and Marx (2016) develop dominant strategy mechanisms that permit estimation of virtual types and implementation via clock auctions. Our paper substantially expands the scope for estimation in mechanism design by applying estimation to a two-sided setup with multi-dimensional types and by showing that under

[6] While we do not do so in this paper, it is conceptually straightforward to accommodate the pursuit of any convex combination of profit and social surplus.

[7] For subsequent generalizations to the case of heterogenous objects, see Ausubel (2006), and Sun and Yang (2009, 2014).

[8] Levin and Skrzypacz (forthcoming) point out that combinatorial clock auctions that allow for simultaneous bids in a stage following the clock stage of the auction have "plausible" inefficient equilibria; indeed, they argue that the efficient equilibrium is "tenuous". Even though our double clock auction is best described as consisting of two phases, they are part of the same clock stage and participating bidders need not be aware of the phase the auction is in. Consequently, our double clock auction is immune to the criticism raised by Levin and Skrzypacz.

reasonable conditions estimation allows the market maker to achieve his goal – either efficiency or profit maximization – in the limit as the market grows.

Third, our paper is also related to the literature on the decentralized foundations of competitive equilibrium, which has focused on showing that the Bayesian equilibrium in simple market mechanisms like the k-double auction converges to the competitive equilibrium as the number of traders grow (see Satterthwaite and Williams, 1989, 2002, Rustichini, Satterthwaite, and Williams, 1994, and Cripps and Swinkels, 2006). [9] When targeting efficiency, one may view our double clock auction as providing a complementary, centralized, and more literal view of the Walrasian auctioneer. In the k-double auction bidders face the complex task of computing the equilibrium strategies.[10] In contrast, in our double clock auction the computational burden is on the shoulders of the market maker, who estimates demand and supply and sets target and reserve prices; agents do not need to have common priors beliefs, their equilibrium strategies are straightforward and the designer may use standard statistical estimation techniques. As our main motivation is to introduce a novel design that can be useful in practice, irrespective of the number of traders, in constructing our double clock auction we followed the commonly held view in market design that the designer should do the heavy computational lifting in the organization of a market, allowing bidders to focus on how they value the assets. Rather than getting rid of the Walrasian auctioneer, we have filled his role with substance.

Finally, the important contributions of McAfee (1992) and Kojima and Yamashita (2014) deserve discussing. McAfee (1992) proposed a dominant strategy double auction for the case of unit demand buyers and unit supply sellers that is an important precursor to our paper. A simplified version of McAfee's double auction sets the quantity traded to be one unit less than the efficient quantity by excluding the least valuable Walrasian trade. It sets a price for buyers equal to the value of the excluded buyer and a price for sellers equal to the cost of the excluded seller.[11] Kojima and Yamashita (2014) have introduced a double auction for a setting with multi-unit demand and supply. Their focus is different from ours, as they are interested in ex post equilibrium in the case of interdependent values. They assume a single dimensional type and that a single crossing condition holds to escape from the impossibility

[9] Other papers on the convergence to competitive equilibrium in the single-unit case include Gresik and Satterthwaite (1989) who looked at optimal trading mechanisms, Yoon (2001) who studied a double auction with participation fees and Tatur (2005), who introduced a double auction with a fixed fee. For the multi-unit case, Yoon (2008) introduced the participatory Vickrey-Clarke-Groves mechanism.

[10] With multi-dimensional types, Jackson and Swinkels (2005) have proven the existence of an equilibrium in distributional strategies, but little is known about the form of the equilibrium strategies.

[11] The general version of McAfee's double auction, which induces the efficient quantity to be traded in some instances, is discussed in footnote 21. Several papers in the management and operations research literature have proposed modifications that expand the possible practical use of McAfee's double auction; see Chu (2009) and the references therein. None of these papers deal with the case of multi-dimensional types and they do not use our approach of introducing a market maker that estimates demand and supply in a double clock setting.

results that plague ex post implementation (e.g., see Jehiel et al., 2006) when two-stage mechanisms as in Mezzetti (2004) are not allowed. The basic idea of the double auction of Kojima and Yamashita is to divide the market into several submarkets and to let trades take place only within a submarket. Information from a different submarket is used to select the reference price in a given submarket and a generalized VCG auction is then used to match demand and supply in the submarket.

The remainder of the paper is organized as follows. Section 2 introduces the setup. In Section 3, we look at efficiency as a benchmark and show that any efficient dominant strategy mechanism that respects the individual rationality constraints runs a deficit on every unit traded. Section 4 illustrates the main ideas with a simple example of our double clock auction with estimation-based tâtonnement (DCA). Section 5 introduces the DCA and derives its properties. Section 6 presents the equivalent direct mechanism, while Section 7 shows how the DCA can be extended to incorporate quantity constraints. In Section 8 we study the asymptotic properties of the DCA, and Section 9 concludes.

2 The Setup

There are N buyers, indexed by $b \in \mathcal{N} = \{1, ..., N\}$, and M sellers, indexed by $s \in \mathcal{M} = \{1, ..., M\}$, of a homogenous good. Let $v_k^b \in [0,1]$ be buyer b's marginal value for the k-th unit of the good and $c_k^s \in [0,1]$ be seller s's cost for producing, or giving up the use of, the k-th unit. Each buyer has a finite maximum demand, or capacity, and each seller has a finite maximum production capacity. Let k_B and k_S be upper bounds on the capacities of buyers and sellers. Then, $K_B = Nk_B$ and $K_S = Mk_S$ are upper bounds on aggregate demand and supply, respectively, and $K = \min\{K_B, K_S\}$ is an upper bound on the quantity that could be traded.

Denote by $\boldsymbol{v}^b = \left(v_1^b, ..., v_{k_B}^b\right)$ the valuation function, or type, of buyer b; $\boldsymbol{c}^s = \left(c_1^s, ..., c_{k_S}^s\right)$ the cost function, or type, of seller s; $\boldsymbol{v} = \left(\boldsymbol{v}^1, ..., \boldsymbol{v}^N\right) = \left(\boldsymbol{v}^b, \boldsymbol{v}^{-b}\right)$ the profile of valuations and $\boldsymbol{c} = \left(\boldsymbol{c}^1, ..., \boldsymbol{c}^M\right) = \left(\boldsymbol{c}^s, \boldsymbol{c}^{-s}\right)$ the profile of costs. We assume diminishing marginal values and increasing marginal costs; that is, for all $b \in \mathcal{N}$, all $k \in \{1, .., k_B - 1\}$, we have $v_k^b \geqslant v_{k+1}^b$ and, for all $s \in \mathcal{M}$, all $k \in \{1, .., k_S - 1\}$, we have $c_k^s \leqslant c_{k+1}^s$.

A buyer b receiving q goods at unit prices $p_1^b, ..., p_q^b$ obtains payoff $\sum_{k=1}^q \left(v_k^b - p_k^b\right)$; a buyer receiving no units and making no payments has zero payoff. Similarly, a seller s selling q goods at prices $p_1^s, ..., p_q^s$ obtains payoff $\sum_{k=1}^q \left(p_k^s - c_k^s\right)$; a seller receiving no payments and selling no units has zero payoff. The payoff functions are common knowledge, but marginal values and marginal costs are private information of each trader.[12]

[12]The upper bounds on traders capacities are common knowledge, but the actual quantities demanded at price 0 and quantities supplied at price 1 are private information.

For every profile of buyers' and sellers' types $\boldsymbol{\theta} = (\boldsymbol{v}, \boldsymbol{c})$ and an equilibrium notion, a double auction mechanism determines an equilibrium allocation profile $\boldsymbol{q}(\boldsymbol{\theta})$, that specifies the quantities $q^b(\boldsymbol{\theta}) \geqslant 0$ and $q^s(\boldsymbol{\theta}) \geqslant 0$ traded by each buyer $b \in \mathcal{N}$ and seller $s \in \mathcal{M}$, and an equilibrium transfer payment profile $\boldsymbol{t}(\boldsymbol{\theta})$, that specifies the payment $t^b(\boldsymbol{\theta})$ from buyer b to the mechanism and the transfer $t^s(\boldsymbol{\theta})$ from the mechanism to seller s.

Let $q_B(\boldsymbol{\theta}) = \sum_{b\in\mathcal{N}} q^b(\boldsymbol{\theta})$ be the total quantity acquired by buyers and $q_S(\boldsymbol{\theta}) = \sum_{s\in\mathcal{M}} q^s(\boldsymbol{\theta})$ be the total quantity given up by sellers. We are interested in mechanisms $\langle \boldsymbol{q}, \boldsymbol{t} \rangle$ that are feasible, individually rational and deficit free.

A mechanism is *feasible* if for every $\boldsymbol{\theta}$, $q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta})$.

Given that the outside option has zero value for every agent, a mechanism satisfies ex post *individual rationality* if for all b, $\boldsymbol{\theta} = \left(\boldsymbol{v}^b, \boldsymbol{\theta}^{-b}\right)$ and for all s, $\boldsymbol{\theta} = (\boldsymbol{c}^s, \boldsymbol{\theta}^{-s})$:

$$\sum_{k=0}^{q^b(\boldsymbol{\theta})} v_k^b - t^b(\boldsymbol{\theta}) \geqslant 0; \qquad t^s(\boldsymbol{\theta}) - \sum_{k=0}^{q^s(\boldsymbol{\theta})} c_k^s \geqslant 0,$$

with the notational convention that $v_0^b = 0$ and $c_0^s = 0$.

The profit a mechanism generates at $\boldsymbol{\theta}$ is $\Pi(\boldsymbol{\theta}) = \sum_{b\in\mathcal{N}} t^b(\boldsymbol{\theta}) - \sum_{s\in\mathcal{M}} t^s(\boldsymbol{\theta})$; a mechanism is *deficit free* if for all $\boldsymbol{\theta}$, $\Pi(\boldsymbol{\theta}) \geqslant 0$.

We will define a flexible class of mechanisms that may target efficiency, maximum profit for the designer, or a convex combination thereof. Each mechanism in our class will satisfy two additional properties whose importance has been recognized equally by the academic literature and design practitioners. The first property is that the mechanism be detail free in the sense of Wilson (1987) and robust in the sense of Bergemann and Morris (2005); that is, it can be specified by the designer without making use of detailed a priori information about agents' types and beliefs, and agents do not need well specified beliefs about the other agents' types in order to bid optimally. Thus, we will use as our equilibrium notion *dominant strategy* equilibrium. The second property that we require is that the mechanism can be run in an open format as a *double clock auction*; that is, it can be run with an ascending clock on the buyers' side and a descending clock on the sellers' side. This implies that the mechanism is privacy preserving; that is, it does not reveal the marginal values or marginal costs of the units that are traded.[13]

3 Efficiency

It is instructive to start with a look at the constraints efficiency imposes on general dominant strategy mechanisms. In our setting, ex post efficiency requires that for all possible type

[13]See Engelbrecht-Wiggans and Kahn (1991), Naor et al. (1999), Ausubel (2004) and Milgrom and Segal (2015) for discussions of the importance of this requirement.

profiles the buyers with the highest marginal valuations trade with the sellers with the lowest marginal costs and that the total quantity traded is $q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta}) = q_W(\boldsymbol{\theta})$, where $q_W(\boldsymbol{\theta})$ is a Walrasian (competitive equilibrium) quantity associated with $\boldsymbol{\theta}$:[14,15]

$$\max\left\{q \in \{0, ..., K\} : v_{(q)} > c_{[q]}\right\} \leqslant q_W(\boldsymbol{\theta}) \leqslant \max\left\{q \in \{0, ..., K\} : v_{(q)} \geqslant c_{[q]}\right\}.$$

By the taxation and revelation principles (see Rochet, 1985, and Myerson, 1979), any dominant strategy mechanism $\langle \boldsymbol{q}, \boldsymbol{t}\rangle$ is strategically equivalent to a "direct" price mechanism $\langle \boldsymbol{q}, \boldsymbol{p}\rangle = \langle\{\boldsymbol{q}, p_k^i(\boldsymbol{\theta}^{-i})\}_{i\in\mathcal{N}\cup\mathcal{M},k=0,...,k_i}\rangle$, with $k_i = k_B$ if $i \in \mathcal{N}$ and $k_i = k_S$ if $i \in \mathcal{M}$, that sets an individualized marginal price schedule for each agent as a function of the other agents' types, lets each agent decide how many units to trade, and has the property that each agent will find it optimal to trade the quantity specified by $\langle \boldsymbol{q}, \boldsymbol{t}\rangle$. The price vector for agent i is $\boldsymbol{p}^i(\boldsymbol{\theta}^{-i}) = \left(p_0^i(\boldsymbol{\theta}^{-i}), .., p_{k_i}^i(\boldsymbol{\theta}^{-i})\right)$, where $p_k^i(\boldsymbol{\theta}^{-i})$ is the price agent i must pay (if a buyer) or must be paid (if a seller) for the k-th unit of the good.[16]

Because the type spaces are smoothly connected, dominant strategy and ex post efficiency can be satisfied if and only if the mechanism is a Groves mechanism (e.g., see Holmström, 1979). Ex post individual rationality and deficit minimization further restrict the mechanism to be a VCG mechanism. Recalling (from footnote 15) that $\theta_{(k)}^{-i}$ is the k-th highest element of the vector $\boldsymbol{\theta}^{-i}$ and $\theta_{[k]}^{-i}$ its k-th lowest element, prices in a VCG mechanism, or two-sided VCG auction, $\langle \boldsymbol{q}, \boldsymbol{p}\rangle$ are defined as follows: for all $b \in \mathcal{N}$ and all $s \in \mathcal{M}$,

$$\boldsymbol{p}^b = \left(0, \theta_{(K_S)}^{-b}, \theta_{(K_S-1)}^{-b}, \theta_{(K_S-2)}^{-b}, ...\right) \quad \text{and} \quad \boldsymbol{p}^s = \left(0, \theta_{[K_B]}^{-s}, \theta_{[K_B-1]}^{-s}, \theta_{[K_B-2]}^{-s}, ...\right).$$

In a VCG mechanism, a buyer acquiring no units pays $p_0^b = 0$ and a seller not selling any units is paid $p_0^s = 0$. To see that $p_1^b = \theta_{(K_S)}^{-b}$ is the negative externality buyer b imposes on the other traders by acquiring her first unit, imagine the mechanism designer collecting all K_S units from the sellers and then efficiently allocating them to the traders (buyers and sellers), buyer b excluded, with the K_S highest marginal values and costs. Since $\theta_{(K_S)}^{-b}$ is the value or cost of the last assigned unit, it is the loss imposed on others if buyer b obtains that unit instead. By the same reasoning, the externality b imposes by acquiring the q-th unit is $\theta_{(K_S+1-q)}^{-b}$.

Similarly, imagine the designer giving the right to own K_B units to the buyers and then efficiently procuring them from the traders, seller s excluded, with the K_B lowest marginal values and costs. The positive externality of seller s on all other traders from selling her first

[14] Ex post efficiency implies feasibility.

[15] Given a vector $\boldsymbol{x}$, we denote by $x_{(i)}$ its i-th highest element and by $x_{[i]}$ its i-th lowest element. Thus, $x_{(q)} = x_{[n+1-q]}$ if the vector contains n elements. We also adopt the notational convention that $v_{(0)} = 1$ and $c_{[0]} = 0$.

[16] A dominant strategy mechanism must be monotonic in the following sense: For all $b \in \mathcal{N}$ and all $\theta^{-b} \in \Theta^{-b}$, where Θ^{-i} is the type space of all agents other than i, $\boldsymbol{v}^b \geqslant \widehat{\boldsymbol{v}}^b$ implies $q^b\left(\boldsymbol{v}^b, \theta^{-b}\right) \geqslant q^b\left(\widehat{\boldsymbol{v}}^b, \theta^{-b}\right)$; for all $s \in \mathcal{M}$ and all $\theta^{-s} \in \Theta^{-s}$, $\boldsymbol{c}^s \leqslant \widehat{\boldsymbol{c}}^s$ implies $q^s\left(\boldsymbol{c}^s, \theta^{-s}\right) \geqslant q^s\left(\widehat{\boldsymbol{c}}^s, \theta^{-s}\right)$.

unit is $p_1^s = \theta_{[K_B]}^{-s}$, the cost or value of the last unit procured when s is excluded, which is saved if seller s sells that unit instead. The externality that s induces when selling the q-th unit is $\theta_{[K_S+1-q]}^{-b}$.[17]

3.1 Deficit on every trade

It is well known from Vickrey's (1961) analysis that the VCG mechanism is not deficit free. To the best of our knowledge, the next theorem is the first to make the stronger claim that in the setting of a market for a homogeneous good the two-sided VCG auction runs a deficit on *each trade*.[18]

For given $\boldsymbol{\theta}$ and Walrasian quantity $q_W(\boldsymbol{\theta})$, let $[\underline{p}_W(\boldsymbol{\theta}), \overline{p}_W(\boldsymbol{\theta})]$ be the Walrasian price gap; that is, let $\underline{p}_W(\boldsymbol{\theta}) = \max\{v_{(q_W(\boldsymbol{\theta})+1)}, c_{[q_W(\boldsymbol{\theta})]}\}$ and $\overline{p}_W = \min\{v_{(q_W(\boldsymbol{\theta}))}, c_{[q_W(\boldsymbol{\theta})+1]}\}$. For any price p in the Walrasian price gap there must be $q_W(\boldsymbol{\theta})$ marginal values and $K_S - q_W(\boldsymbol{\theta})$ marginal costs at least as high as p; that is, $\overline{p}_W = \theta_{(K_S)}$. There must also be $K_B - q_W(\boldsymbol{\theta})$ marginal values and $q_W(\boldsymbol{\theta})$ marginal costs at least as low as p; that is, $\underline{p}_W = \theta_{[K_B]} = \theta_{(K_S+1)}$.

Theorem 1. (Deficit of at least $\overline{p}_W(\boldsymbol{\theta}) - \underline{p}_W(\boldsymbol{\theta})$ on each unit traded in the two-sided VCG.) *In the two-sided VCG auction, $\overline{p}_W(\boldsymbol{\theta}) = \theta_{(K_S)}$ is the lowest price paid to any seller for a unit sold and $\underline{p}_W(\boldsymbol{\theta}) = \theta_{(K_S+1)}$ is the highest price paid by any buyer for a unit bought. A dominant strategy, ex post individually rational mechanism that implements the ex post efficient (Walrasian) allocation does not generate a positive profit for any type profile and generates a negative profit as long as $q_W(\boldsymbol{\theta}) > 0$ and $\theta_{(K_S)} > \theta_{(K_S+1)}$.*

Proof. Suppose efficiency requires s to sell quantity $q^s(\boldsymbol{\theta})$ at type profile $\boldsymbol{\theta}$. Since the VCG price vector of seller s, $\boldsymbol{p}^s = (p_0^s, p_1^s, ..., p_k^s, ...)$ is decreasing in k for $k \geqslant 1$, the lowest price paid to s on a unit sold is $p_{q^s(\boldsymbol{\theta})}^s(\boldsymbol{\theta}^{-s}) = \boldsymbol{\theta}_{[K_B+1-q^s(\boldsymbol{\theta})]}^{-s}$. It must be $c_{q^s(\boldsymbol{\theta})}^s \leqslant p_{q^s(\boldsymbol{\theta})}^s(\boldsymbol{\theta}^{-s})$, since s sells $q^s(\boldsymbol{\theta})$ units and hence has marginal cost below $\boldsymbol{\theta}_{[K_B+1-q^s(\boldsymbol{\theta})]}^{-s}$ for at least $q^s(\boldsymbol{\theta})$ units. This implies that $\boldsymbol{\theta}_{[K_B+1-q^s(\boldsymbol{\theta})]}^{-s} \geqslant \boldsymbol{\theta}_{[K_B+1]} = \boldsymbol{\theta}_{(K_S)}$, where the equality holds because the vector $\boldsymbol{\theta}$ contains $K_B + K_S$ elements. This shows that $p_{q^s(\boldsymbol{\theta})}^s(\boldsymbol{\theta}^{-s}) \geqslant \boldsymbol{\theta}_{(K_S)}$.

Now suppose efficiency requires b to buy quantity $q^b(\boldsymbol{\theta})$ at type profile $\boldsymbol{\theta}$. Since the VCG price vector of buyer b is increasing, the highest price paid on a unit acquired is $p_{q^b(\boldsymbol{\theta})}^b\left(\boldsymbol{\theta}^{-b}\right) = \boldsymbol{\theta}_{(K_S+1-q^b(\boldsymbol{\theta}))}^{-b}$. It must be $v_{q^b(\boldsymbol{\theta})}^b \geqslant p_{q^b(\boldsymbol{\theta})}^b\left(\boldsymbol{\theta}^{-b}\right)$, since b buys $q^b(\boldsymbol{\theta})$ units and hence has marginal value above $\boldsymbol{\theta}_{(K_S+1-q^b(\boldsymbol{\theta}))}^{-b}$ for at least $q^b(\boldsymbol{\theta})$ units. This implies that $\boldsymbol{\theta}_{(K_S+1-q^b(\boldsymbol{\theta}))}^{-b} \leqslant \boldsymbol{\theta}_{(K_S+1)}$ and shows that $p_{q(\boldsymbol{\theta})}^b\left(\boldsymbol{\theta}^{-b}\right) \leqslant \boldsymbol{\theta}_{(K_S+1)}$.

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[17] Note that each buyer's unit price is increasing; the price on the $(q+1)$-th unit is at least as high as the price on the q-th unit. Similarly, each seller's unit price is decreasing; the price of the q-th unit sold is at least as high as the price on the $(q+1)$-th unit.

[18] Vickrey seems to have been aware only of the aggregate deficit; he wrote (1961, p. 13, emphasis added): "The basic drawback to this scheme is, of course, that the marketing agency will be required to make payments to suppliers in an amount that exceeds, *in the aggregate*, the receipts from purchasers...".

Thus, we conclude that $p^s_{q^s(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-s}\right) \geqslant \boldsymbol{\theta}_{(K_S)} \geqslant \boldsymbol{\theta}_{(K_S+1)} \geqslant p^b_{q(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-b}\right)$; seller s is paid a price on any unit sold at least as high as the price paid on any unit acquired by buyer b.

The second statement of the theorem follows from the first statement, payoff equivalence (see Holmström, 1979), and the non-trading agents obtaining their zero, outside-option payoff, which imply that the VCG mechanism minimizes the deficit among all efficient, dominant strategy, ex post individual rationality mechanisms. □

Only under very special type profiles with ties that guarantee that all efficient trades can be completed at a uniform price will a two-sided VCG auction balance the budget. Suppose for example that there are two buyers demanding one unit each, Bill with a value of 3 and Nancy with a value of 2, and two sellers with one unit each, Roger with a cost of 2 and Susan with a cost of 1. There are two possible Walrasian quantities, $q_W \in \{1, 2\}$; if $q_W = 1$ is chosen, then the VCG mechanism stipulates that the only trade is between Bill and Susan at the price of 2 and there is no deficit.

The intuition for Theorem 1 is subtle but clear. From the point of view of buyers, the two-sided VCG auction allocates K_S units to the agents with the K_S highest types in the ordered list $\boldsymbol{\theta} = (\theta_{(1)}, .., \theta_{(K_S)}; \theta_{(K_S+1)}, ..., \theta_{(K_B+K_S)})$. If buyer b acquires a positive number of units under efficiency, it must be the case that she prevents as many units from being obtained by other agents that would obtain these units under efficiency if b were not there. Consequently, with b present, the values or costs of these units belong to the bottom K_B elements of $\boldsymbol{\theta}$ and constitute the social opportunity cost b exerts. Likewise, from the point of view of sellers the two-sided VCG auction procures K_B units from the agents with the K_B lowest types in the ordered list $\boldsymbol{\theta}$. If seller s procures a positive number of units under efficiency, it must be the case that her presence crowds out an equal number of units from being procured from other agents. Consequently, the values or costs of the units that s crowds out belong to the top K_S elements in $\boldsymbol{\theta}$ and represent the social value s's presence adds. Taken together, buyers pay unit prices on units traded under efficiency that reflect elements from the bottom K_B entries in $\boldsymbol{\theta}$ while sellers are paid unit prices for units traded under efficiency that reflect elements from the top K_S entries in $\boldsymbol{\theta}$.

3.2 Possible Remedies and Challenges

Following the same logic used in the proof of Theorem 1, one can show that in the two-sided VCG auction the lowest price a buyer pays is at least as high as $\boldsymbol{\theta}_{(K_S-k_B)}$ and the highest price a seller is paid is at most $\boldsymbol{\theta}_{[K_B-k_S]}$. Thus, one might be tempted to "fix" the deficit problem by adjusting prices upwards for buyers and downwards for sellers. For example, the following "adjusted VCG" prices would guarantee that for each unit buyers pay at least as much as sellers must be paid; for all $b \in \mathcal{N}$ and all $s \in \mathcal{M}$,

$$\boldsymbol{p}^b = \left(0, \theta^{-b}_{(K_S-k_B)}, ..., \theta^{-b}_{(K_S-k_B+1-k)}, ...\right) \text{ and } \boldsymbol{p}^s = \left(0, \theta^{-s}_{[K_B-k_S]}, ..., \theta^{-s}_{[K_B-k_S+1-k]}, ...\right).$$

There is no guarantee, however, that under such a pricing scheme demand equals supply. Suppose there are two buyers with valuations $\boldsymbol{v}^{b_1} = (6,3), \boldsymbol{v}^{b_2} = (5,4)$ and two sellers with costs $\boldsymbol{c}^{s_1} = (1,7), \boldsymbol{c}^{s_2} = (2,8)$. The "adjusted VCG" prices are $\boldsymbol{p}^{b_1} = \boldsymbol{p}^{b_2} = (7,8), \boldsymbol{p}^{s_1} = (3,2), \boldsymbol{p}^{s_2} = (3,1)$ and it is immediate to see that at such prices each buyer demands zero units and each seller supplies one unit.

This example highlights the first conceptual challenge one must overcome in designing deficit free, dominant strategy, approximately efficient double auctions. In contrast to the one-sided setting in which the quantity to be traded is known at the outset, satisfying the feasibility constraint is not trivial in our two-sided environment, as one needs to make sure that the quantity that sellers want to sell is equal to the quantity that buyers want to buy. This means that there needs to be some link between the buyers and sellers' side of the double auction mechanism, and care must be taken to guarantee that such a link does not lead to a mechanism that violates the dominant strategy property.

Theorem 1 naturally suggests reserve prices that act as barriers between what buyers pay and what sellers collect as an important ingredient in the design of deficit free, individually rational, double auctions.[19] The second conceptual challenge to overcome in such a design is to keep the efficiency losses created by reserve prices small, and ideally to guarantee that they disappear as the market grows. For example, consider a mechanism with an arbitrary, exogenously set, posted price r, as in Hagerty and Rogerson (1987), with the added feature that the quantity traded is equal to the quantity demanded or supplied on the short side of the market, and with an Ausubel auction run on the long side of the market to satisfy the feasibility constraint. Letting q be the minimum of the quantity demanded and supplied at r, the elements of the unit price vector buyer b faces, on any unit $k > 0$, are $p^b_k = \max\{r, \mathbf{v}^{-b}_{(q+1-k)}\}$ and $p^s_k = \min\{r, \mathbf{c}^{-s}_{[q+1-k]}\}$. Such a mechanism is dominant strategy incentive compatible, but may result in dramatic efficiency losses even in thick markets if the exogenously posted price differs from the Walrasian price.[20]

The third challenge is to guarantee that no trader has an incentive to manipulate the choice of the endogenous reserve prices and the determination of the quantity traded. This is a much more difficult challenge in our setting with multi-unit demand and supply and

[19] With a single unit-demand buyer and a single unit-supply seller, Hagerty and Rogerson (1987) reached a strong conclusion about the role of reserve prices. They showed that posted-price mechanisms, in which a price is posted and trade occurs if and only if buyer and seller agree to trade, are "essentially" the only dominant strategy, ex post individually rational, budget balanced mechanisms.

[20] A similar challenge applies to the case, also considered in this paper, in which the designer's goal is to obtain maximum profit; with exogenously set reserve prices there is no guarantee that the profit loss is small and disappears as the market grows.

multi-dimensional traders types, than in the case of single-dimensional agent types studied in most of the literature. In the case of single dimensional types with unit-demand buyers and unit-supply sellers, McAfee (1992) proposed a dominant strategy mechanism which in a simplified form works as follows. The quantity traded is reduced by one unit below the efficient quantity by excluding the *least valuable Walrasian trade*. The price for buyers is equal to the value of the excluded buyer and the price for sellers is equal to the cost of the excluded seller.[21] An alternative interpretation of the mechanism is that it first computes the Walrasian quantity under the assumption that the quantity demanded is shifted up by one unit. Then it sets the quantity traded to be the shifted Walrasian quantity minus one unit, and uses as the reserve price for buyers the highest value among the non-trading buyers. Finally, it runs an efficient auction for sellers.

Excluding the least valuable Walrasian trade would not work in our multi-dimensional setting, as the buyer and seller associated with that trade (and hence setting the trading prices) may trade some units after the last trade is excluded. They would then have an incentive to manipulate the price on their side of the market. For the same reason, shifting demand by $k = \max\{k_B, k_S\}$ units would not work; the buyer setting the reserve price may trade some units and have an incentive to manipulate the price. Another possibility would be to exclude the *least valuable Walrasian traders*; that is, the buyer with the lowest marginal value and the seller with the highest marginal cost for their first unit, among those that would trade a positive quantity under the efficient allocation. One could then set reserve prices for buyers and sellers respectively equal to the excluded buyer's marginal value and the excluded seller's marginal cost of their first unit. At such prices demand need not equal supply, but one could set the quantity traded to be the short side quantity, trade at the short side reserve price on the short side, and run an Ausubel auction on the long side with the long side reserve price. This mechanism, however, is not dominant strategy incentive compatible with multi-dimensional types. This is because an agent who trades a positive quantity may be able to change the excluded trader on her side of the market to her benefit. For example, by reducing supply a seller may induce a change in the excluded seller, with the previously excluded seller now trading some units, thereby increasing the sellers' reserve price.

[21]The general version of McAfee's mechanism yields efficient trading under some type realizations. Let p^W be a linear combination of the highest buyers' value and the lowest sellers' cost among agents who would not trade under a Walrasian allocation. Whenever p^W is in the Walrasian price gap, it is the price set by McAfee's mechanism and it leads to efficient trading. When p^W is outside the Walrasian price gap, the simplified version of the mechanism applies.

4 Estimation-Based Tâtonnement: An Illustration

Our approach to these challenges is to introduce a double clock auction that uses an estimation-based tâtonnement and consists of a price discovery and an allocation phase. In this section, we describe the main ideas underlying our approach and illustrate the working of our double clock auction (DCA) with a simple example. The formal description of our mechanism and its properties are then provided in Section 5.

In each phase of our DCA, there is an ascending-price clock for buyers and a descending-price clock for sellers.[22] We denote by p_t^B and p_t^S the buyers' clock and sellers' clock prices at time t. The only action an agent may take in each phase is to reduce her demand or supply by one or more units. Let d_t^b be the quantity demanded by buyer $b \in \mathcal{N}$ at time t, $D_t = \sum_{b \in \mathcal{N}} d_t^b$ be the total quantity demanded at time t, s_t^s be the quantity supplied by seller $s \in \mathcal{M}$ at time t, and $S_t = \sum_{s \in \mathcal{M}} s_t^s$ be the total quantity supplied at time t. Demand and supply reductions are irrevocable, and once the demand of a buyer or the supply of a seller has reached zero, the agent becomes inactive and exits the double clock auction.

The first phase of the DCA, called the *discovery phase*, uses information from traders that have become inactive to set reserve prices; that is, a price floor r^B for buyers and a price ceiling r^S for sellers. Two key elements of our design of the discovery phase are the use of *target prices* and an *estimation-based tâtonnement process*. The target prices are the announced prices at which the clocks temporarily stop in a given round if no exit has occurred in the round. To avoid manipulation opportunities, the target prices only vary with the demand and supply revealed by agents who have become inactive in previous rounds. Each round lasts one unit of time. We denote by p_t^{BT} and p_t^{ST} the round t target prices for the buyers' and sellers' clock, and by I_t^B and I_t^S the sets of buyers and sellers that have dropped out and become inactive in rounds prior to round t; that is, we have $b \in I_t^B$ if and only if $d_{t-1}^b = 0$ and $s \in I_t^S$ if and only if $s_{t-1}^s = 0$. Let $I_t \equiv I_t^B \cup I_t^S$.

The guiding principle of our price adjustment process in the discovery phase is Walras' tâtonnement. An important novelty of our approach is that the tâtonnement process is driven by *estimated* excess demand, rather than the "true" (or revealed) excess demand. We decrease the sellers' clock price when estimated excess demand is negative at the standing prices, increase the buyers' clock price when estimated excess demand is positive and move both clock prices at a synchronized speed so that they would simultaneously reach their target prices whenever estimated excess demand is zero at the standing clock prices. Estimated market demand in round t, denoted $\widehat{D}_t(p)$, is restricted to be a decreasing function of price, that is, $\widehat{D}_t(p) : [0,1] \to \mathbb{R}$, while estimated market supply, $\widehat{S}_t(p)$, is an increasing function

[22] It is simpler to describe our DCA if we allow the two clocks to move continuously; apart for having to pay additional care to the case of ties, nothing substantial would change if we had the clock move by small discrete increments.

$\widehat{S}_t(p) : [0,1] \rightarrow \mathbb{R}$. Thus, $\widehat{Z}_t\left(p_t^B, p_t^S\right) = \widehat{D}_t\left(p_t^B\right) - \widehat{S}_t\left(p_t^S\right)$ is estimated excess demand in round t at buyers' price p_t^B and sellers' price p_t^S. Assuming differentiability of estimated demand and supply, define the functions $\widehat{MR}_t(\cdot)$ and $\widehat{MC}_t(\cdot)$ as the estimated marginal revenue and marginal procurement cost of an increase in quantity traded:[23]

$$\widehat{MR}_t(p) = p + \frac{\widehat{D}_t(p)}{\frac{\partial \widehat{D}_t(p)}{\partial p}} \quad \text{and} \quad \widehat{MC}_t(p) = p + \frac{\widehat{S}_t(p)}{\frac{\partial \widehat{S}_t(p)}{\partial p}}. \tag{1}$$

To eliminate any incentive to manipulate estimated demand and supply, the only data the designer can use to derive them are the marginal values and costs of buyers and sellers in I_t, the set of traders that have become inactive in rounds prior to t. There are no restrictions on the procedure used to obtain the estimated demand and supply functions (indeed they could even be arbitrary guesses), as long as they are not related to information provided by active agents. For concreteness and simplicity, in the example below we will use ordinary least squares to regress quantity demanded and supplied on price, assuming that all remaining active agents demand and supply at full capacity. For the convergence results of Section 8 and their proofs, on the other hand, we will use a minimum distance estimation procedure.

Once the reserve prices have been determined, information from all traders can be used without introducing manipulation incentives to set the aggregate quantity traded to be equal to the minimum of the true demand at the buyers' reserve price and the true supply at the sellers' reserve price, with true demand and true supply being defined as total demand and supply indicated by the remaining active agents. Then the discovery phase ends and the second phase of the DCA, called the *allocation phase*, begins, in which an Ausubel auction starting at r^B is run for buyers and a reverse Ausubel auction starting at r^S is run for sellers. Thus, all units are traded at the reserve prices if true excess demand is zero at these prices. Otherwise, all units on the short side of the market at these prices are traded at the reserve price, while the unit prices on the long side are given by the VCG unit prices, bounded by the reserve price. As we shall prove, active agents can only manipulate the total quantity traded in an unprofitable way. For instance, when buyers are on the short side, by misrepresenting her true demand an active buyer could raise total quantity traded at the cost of acquiring units valued below the reserve price, or reduce total quantity traded at the cost of not obtaining units valued more than the reserve price. When buyers are on the long side, an active buyer can only affect quantity traded without becoming inactive by reducing demand, but this would not affect the buyers' reserve price and would come at the cost of not obtaining units at a price below marginal value.

Our design offers many degrees of freedom. It can be used to pursue the objective of efficiency subject to avoiding a deficit or the objective of maximizing profit. Intermediate

[23] Since estimated demand and supply are monotone functions of price, they are differentiable almost everywhere.

goals between efficiency and profit could also be pursued.[24] If the designer's goal is efficiency, the estimates are used to set reserve prices as close as possible to the estimated Walrasian price. If the target is profit, the reserve prices are set so as to be as close as possible to the prices that equalize estimated marginal revenue and marginal procurement cost. With a small number of traders, such estimates might be inaccurate even when they are carefully chosen, but the dominant strategy property and all the other desiderata will be satisfied. We will show in Section 8 that under mild assumptions minimum distance estimates converge to the true functions as the number of traders increases.

Our design can also accommodate quantity constraints, such as a cap on the number of units a subgroup of buyers may acquire in total. In practical applications this is important, as constraints like these arise for a number of reasons, antitrust concerns being one of them.

An Example. The following example illustrates the working of our DCA. There are eight buyers who demand at most three units of a homogeneous good and fourteen sellers who each have maximum capacity for producing two units.

Buyer 1	90	86	71
Buyer 2	88	58	37
Buyer 3	*84*	77	25
Buyer 4	66	**54**	46
Buyer 5	62	42	0.7
Buyer 6	50	0.6	0.5
Buyer 7	31	0.4	0.3
Buyer 8	19	0.2	0.1

Table 1: Buyers' marginal valuations.

Tables 1 and 2 contain agents' valuations and costs. For example, buyer 3 is willing to pay 84 for the first unit, 77 for the second and 25 for the third unit, while Seller 1's marginal costs for producing the first and second unit are 1 and 33, respectively.[25]

In Figure 1, panel (a) depicts market demand and supply and panel (b) the associated marginal revenue and marginal procurement cost.[26] The efficient quantity traded is 10, while 4 is the quantity that maximizes the profit of a price-posting market maker that knows market demand and supply.[27] The Walrasian price gap is $[53, 54]$ as indicated in bold in Tables 1

[24]It would be straightforward to maximize any convex combination of profit and social surplus, by using a variant of Ramsey pricing and equating the appropriately weighted marginal revenue and marginal procurement cost.

[25]To simplify the computations, we have avoided ties in the valuations and costs. To improve readability, we have blown up to $[0, 100]$ the interval from which valuations and costs are selected.

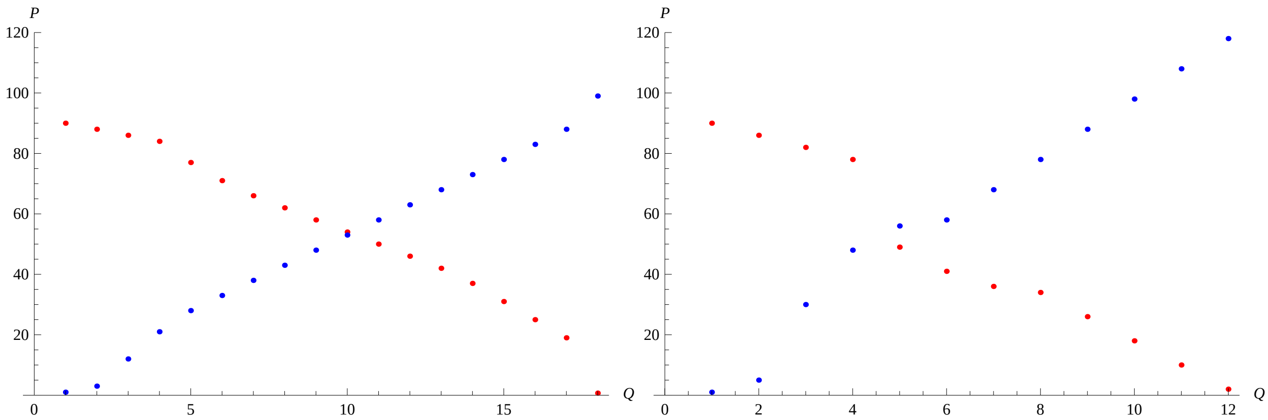
[26]Marginal revenue for the kth unit is $kv_{(k)} - (k-1)v_{(k-1)}$. Analogously, the marginal cost for procuring the kth unit is $kc_{[k]} - (k-1)c_{[k-1]}$.

[27]Ex post profit maximization with posted prices is a natural benchmark. It fares prominently in the

Seller 1	1	33		Seller 8	48	99.3
Seller 2	3	58		Seller 9	**53**	99.4
Seller 3	12	100		Seller 10	68	99.5
Seller 4	*21*	63		Seller 11	73	99.6
Seller 5	28	99		Seller 12	78	99.7
Seller 6	38	99.1		Seller 13	83	99.8
Seller 7	43	99.2		Seller 14	88	99.9

Table 2: Sellers' marginal costs.

and 2. The profit maximizing prices, highlighted in italics in the tables, are 84 for buyers and 21 for sellers, resulting in a total profit of 252.



(a) Demand and supply (b) Marginal revenue and cost

Figure 1: The efficient (panel a) and the profit maximizing (panel b) quantities.

Efficiency targeting DCA. Assuming sincere bidding, which is, as we shall prove, a dominant strategy equilibrium, Table 3 illustrates the working of the efficiency targeting DCA. There are several rounds, each round taking one unit of time to complete. $|I_t^B|$ and $|I_t^S|$ are the cumulative numbers of buyers and sellers who have dropped out before round t, $\widehat{Z}_t$ is estimated excess demand at the beginning of round t, p_{t-1}^B and p_{t-1}^S are the clock prices at the end of round $t-1$ and the beginning of round t. The last column specifies which clock is moving in the round, either the buyers' clock (B), the sellers' clock (S) or both. p_t^{BT} denotes the buyers' target price in a round t in which the buyers' clock moves; p_t^{ST} denotes the sellers' target price in a round t in which the sellers' clock moves. If a clock moves in a round, it

operations research and computer science literature, even though with a finite number of traders no incentive compatible mechanism can achieve it, as traders have an incentive to misreport their valuations.

moves continuously so as to reach its target in a unit of time. If a trader drops out in the round, all clocks stop and the next round starts once a full unit of time has elapsed.

The discovery phase of the DCA starts in round 1 with the buyers' clock at $p_0^B = 0$ and the sellers' clock at $p_0^S = 100$. Initially buyers are assigned a demand of $k_B \geqslant 3$ units and sellers a supply of $k_S \geqslant 2$ units each.

As maximum total demand and supply are 24 and 28, respectively, we take $\widehat{D}_1(p) = 24 - 6p/25$ and $\widehat{S}_1(p) = 28p/100$ as the initial estimates, before any agents have exited, of demand and supply at price $p \in [0, 100]$. For every round, we have highlighted in bold the event that started the round by triggering the end of the previous round; this is either one additional trader that dropped out, or a target price that was reached by a clock.

Efficiency Targeting DCA								
round	$\lvert I_t^B\rvert$	$\lvert I_t^S\rvert$	p_{t-1}^B	p_{t-1}^S	p_t^{BT}	p_t^{ST}	$\widehat{Z}_t$	moving clock(s)
1	0	0	0	100		85.71	-4	S
2	0	**1**	0	88		58.28	-2.5	S
3	0	**2**	0	83		78.70	-0.62	S
4	0	2	0	**78.70**	29.40	29.40	0	BOTH
5	0	**3**	0.42	78	5.95		1.33	B
6	0	3	**5.95**	78	36.95	36.95	0	BOTH
7	0	**4**	9.73	73	14.78		1.21	B
8	0	4	**14.78**	73	41.76	41.76	0	BOTH
9	**1**	4	19	68.12	44.78		2.06	B
10	**2**	4	31	68.12		62.76	-1.11	S
11	2	**5**	31	68	31.30		0.04	B
12	2	5	**31.30**	68	55.36	55.36	0	BOTH
13	**3**	5	50	58.17		52.17	-1.36	S
14	3	**6**	50	53		51.90	-0.23	S
15	3	6	50	**51.90**	51.20	51.20	0	BOTH
16	3	6	**51.20**	**51.20**			0	END

Table 3: Discovery phase of the efficiency targeting double clock auction.

In round 1, all agents are active at the starting clock prices of $p_0^B = 0$ and $p_0^S = 100$. Because $\widehat{S}_1(100) = 28 > 24 = \widehat{D}_1(0)$, estimated excess demand at the current clock prices is $\widehat{Z}_1(0, 100) = -4$. Consequently, only the sellers' clock moves; the sellers' target price is $p_1^{ST} = 85.71$, which is the price that makes estimated excess demand $\widehat{D}_1(0) - \widehat{S}_1(p_1^{ST})$ equal to zero. When the sellers' clock reaches 88, seller 14 exits, which brings the clock to a halt and marks the end of round 1. The DCA then regresses the vector of quantities $\boldsymbol{y} = (28, 27, 27, 26)$ onto a constant and onto the prices $\boldsymbol{p} = (99.9, 99.89, 88, 87.99)$, so as to keep track of total supply immediately before and after a reduction in supply due to seller 14; we use 0.01 as the price decrement. This results in the re-estimated supply $\widehat{S}_2(p) = 19.098 + 0.0841p$.[28] Because estimated excess demand is still negative at the standing prices $p_1^B = 0$ and $p_1^S = 88$ (it is $\widehat{Z}_2 = -2.5$ as shown in the second-to-last column), only the sellers' clock is restarted in round 2 with a new target price of $p_2^{ST} = 58.28$, determined by solving $\widehat{D}_2(0) - \widehat{S}_2(p_2^{ST}) = 0$. The sellers' clock stops and round 2 ends at $p_2^S = 83$ with the exit of seller 13. Supply is estimated anew. Only the sellers' clock is restarted in round 3, just like in the previous round, because estimated excess demand is still negative at the standing prices. The new sellers' target price is $p_3^{ST} = 78.70$. Observe that round 3 ends when the sellers' clock reaches the target price $p_3^{ST} = 78.70$ without any additional exit. At this price, estimated supply is equal to estimated demand at the standing buyers' clock price of 0. No additional information is gleaned from the agents, as no exit has occurred. Since estimated excess demand is zero both clocks will move simultaneously in round 4, with synchronised speeds and the common target price of $p_4^{BT} = p_4^{ST} = p_4^T = 29.4$, which is the price that solves $\widehat{D}_4(p_4^T) - \widehat{S}_4(p_4^T) = 0$. Round 4 ends when the sellers' clock reaches 78, at which point seller 12 drops out. The newly estimated supply indicates that estimated excess demand is positive at the current clock prices, so only the buyers' clock moves in round 5; it stops when it reaches the buyers' target price of 5.95, after which both clocks move again in synchronised speeds in round 6.

The discovery phase ends after three more exits on each side of the market when both clocks reach the target price of 51.20, which becomes the reserve price in the allocation phase.[29] Because true total demand at 51.20 is 10 while true market supply is only 9, sellers are on the short side. The equilibrium quantity traded is thus 9, which is one unit short of the efficient quantity. Consequently, the remaining eight active sellers trade at the reserve price of 51.20, with seller 1 being the only one producing two units. The remaining five active buyers compete in an Ausubel auction with a reserve of 51.20 for the 9 units. In the equilibrium of this auction, buyer 1 clinches three units, paying the reserve of 51.20 on each of the first two units and 54 for the third. Buyers 2 and 3 each acquire two units at the unit prices of 51.20 for the first and 54 for the second. Buyer 4 clinches one unit at the reserve price, while buyer 5 clinches one unit at the price of 54, which is the price at which buyer 4 reduces demand from two to one. Accordingly, the revenue of the market maker is 11.2, which is four times the difference between 54 and 51.2.

While there is no guarantee that the quantity traded will always be close to the efficient quantity, we will show that under mild regularity conditions in large economies the loss in social surplus of the efficiency targeting DCA converges to zero.

[28] An analogous procedure, with the data from the inactive traders as input, is used to re-estimate supply and demand each time a new seller or buyer exits. Note that $\widehat{D}_2(p) = \widehat{D}_1(p)$ as no buyer exits in the first round.

[29] The discovery phase always ends in finitely many steps when the number of agents is finite. It ends at the earliest of two events: The target prices are reached by both clocks or the number of active agents on one side of the market drops to zero.

Profit targeting DCA								
round	$\lvert I_t^B\rvert$	$\lvert I_t^S\rvert$	p_{t-1}^B	p_{t-1}^S	p_t^{BT}	p_t^{ST}	$\widehat{Z}_t$	moving clock(s)
1	0	0	0	100		85.71	-4	S
2	0	**1**	0	88		58.28	-2.5	S
3	0	**2**	0	83		78.70	-0.62	S
4	0	2	0	**78.70**	64.70	-29.80	0	BOTH
5	0	**3**	0.42	78	5.95		1.33	B
6	0	3	**5.95**	78	68.48	-4.78	0	BOTH
7	0	**4**	9.73	73	14.78		1.21	B
8	0	4	**14.78**	73	70.88	8.04	0	BOTH
9	**1**	4	19	68.12	44.78		2.06	B
10	**2**	4	31	68.12		62.76	-1.11	S
11	2	**5**	31	68	31.30		0.04	B
12	2	5	**31.30**	68	120.38	21.17	0	BOTH
13	**3**	5	50	58.17		52.17	-1.36	S
14	3	**6**	50	53		51.90	-0.23	S
15	3	6	50	**51.90**	110.10	16.33	0	BOTH
16	3	**7**	56.58	48	61.45		0.60	B
17	3	7	**61.45**	48	111.00	18.65	0	BOTH
18	**4**	7	62	47.68		40.73	-1.45	S
19	4	**8**	62	43	64.36		0.33	B
20	4	8	**64.36**	43	99.14	20.30	-0.01	S
21	4	**9**	64.36	38	78.39		1.95	B
22	**5**	9	66	38	67.99		0.33	B
23	5	9	**67.99**	38		22.60	-0.01	S
24	5	**10**	67.99	28	85.24		2.80	B
25	**6**	10	84	28		23.92	-0.91	S
26	6	10	84	**23.92**			-0.02	END

Table 4: Discovery phase of the profit targeting double clock auction.

Profit targeting DCA. Table 4 illustrates the working of the profit targeting DCA. Like in the efficiency targeting version, the discovery phase starts with the buyers' clock at $p_0^B = 0$, the sellers' clock at $p_0^S = 100$, estimated demand $\widehat{D}_1(p) = 24 - 6p/25$ and estimated supply $\widehat{S}_1(p) = 7p/25$. Each time a trader exits, demand and supply are re-estimated. The first three rounds proceed as in the efficiency targeting version; since estimated excess demand is negative, only the sellers' clock moves and the sellers' target price is set at the point at which estimated excess demand is zero.

At the beginning of round 4, estimated excess demand is zero; both clocks will move simultaneously in this round. Unlike in the efficiency targeting version, the DCA now selects two different target prices, p_4^{BT} for buyers and p_4^{ST} for sellers. These are the prices at which estimated excess demand is zero *and* estimated marginal revenue equals estimated marginal procurement cost. As estimated supply is $\widehat{S}_4(p) = 12.7376 + 0.14311p$ and estimated demand is $\widehat{D}_4(p) = 24 - 0.24p$, it turns out that the sellers' target price is negative, $p_4^{ST} = -29.80$. This is an artefact of using a linear regression estimation procedure without imposing a non-negativity constraint on prices.[30] Round 4 ends with the sellers' clock reaching 78, at which point seller 12 drops out. The DCA continues, with the only difference relative to the efficiency targeting version being that when estimated excess demand is zero (i.e., in rounds 6, 8, 12, 15, and 17) different target prices are set for buyers and sellers, so as to equate estimated demand with estimated supply and estimated marginal revenue with estimated marginal procurement cost.[31]

The discovery phase ends with two active buyers and three active sellers and the target prices of 84 and 23.92. These prices become the reserve prices in the allocation phase. Because market demand at 84 is three while aggregate supply at 23.92 is four, all buyers trade at the reserve of 84 with buyer 1 acquiring two units and buyer 2 one. The remaining active sellers compete in a reverse Ausubel auction with a reserve of 23.92 for the right to supply one of the three units demanded each. This results in a price of 21 for sellers 1, 2, and 3 and a revenue for the market maker of 189, which is 75% of the profit maximum of 252; the quantity traded falls one trade short of the ex post profit maximizing quantity of four.[32] We will show that under mild regularity conditions the profit targeting DCA achieves maximum profit extraction if the market is large.

5 The Double Clock Auction

We now provide the formal description of the DCA and prove that it is feasible, deficit free, ex post individually rational and dominant strategy incentive compatible.

We assume *no bid information* is given to traders either in the discovery or the allocation phase; agents are not even told whether the mechanism is in the discovery phase or in the allocation phase. All the information that is revealed to the agents is the standing clock price on their side of the market and whether the auction on their side is still ongoing. There are two reasons for revealing no bid information to the agents. First, it makes the bidding environment straightforward; much like in the Walrasian analysis of competitive markets, all

[30] Using a more sophisticated estimation procedure would help targeting maximal profit.

[31] Additionally, if estimated marginal revenue at the current buyers' clock price exceeds estimated marginal procurement cost at the current sellers' clock price, the standing clock prices become the target prices irrespective of the sign of estimated excess demand. Like in the efficiency targeting version, the discovery phase ends when both clocks reach their target prices, or when there are no active traders left in one side of the market.

[32] It is a coincidence that the reserve prices of the profit targeting DCA coincide with the prices set by the profit maximizing price-posting mechanism. Since the reserve prices correspond to the marginal value and marginal cost for the first unit of the last buyer and seller that have exited the discovery phase of the DCA, total quantity traded is less than in the profit maximizing price-posting mechanism.

information that an agent needs to make optimal decisions is contained in the prices. Second, as we will show, it endows the mechanism with the desirable feature that sincere bidding by all agents, as defined next, is a dominant strategy equilibrium.[33]

We say that an agent engages in *sincere bidding* if she expresses her quantity demanded or supplied truthfully, that is, if she stays active on a unit until the clock price reaches her value or cost for that unit and reduces activity – either demand or supply – by one unit when the clock price exceeds her value (falls below her cost) for that unit. Formally, buyer b bids sincerely if for any price p_t^B at time t her demand is d_t^b such that $v_{d_t^b}^b \geqslant p_t^B \geqslant v_{d_t^b+1}^b$ and seller s bids sincerely if for any price p_t^S at time t her supply s_t^s is such that $c_{s_t^s}^s \leqslant p_t^S \leqslant c_{s_t^s+1}^s$.

The discovery phase proceeds in several rounds beginning one unit of time apart; round t starts at time $t-1$ and ends at time t.[34] Each round begins in an *estimation state*, during which time stops and no clock moves. After the estimation state, either the discovery phase ends or the state in the round transitions into one of three different states: the *buyers' clock*, *sellers' clock* and *double clock state*.

Discovery Phase: The Estimation State. In the estimation state at the beginning of round t, the clock prices for buyers and sellers are the prices p_{t-1}^B and p_{t-1}^S inherited from the previous round; in round 1, we have $p_0^B = 0$ and $p_0^S = 1$. The estimation state is characterized by two sets of functions. The first set contains estimated demand and supply; that is, the functions $\widehat{D}_t(\cdot), \widehat{S}_t(\cdot)$ (and the associated marginal revenue and cost, see (1)). The second set contains the strictly increasing functions $\sigma_t^B(p) : [0,1] \to \mathbb{R}$ and $\sigma_t^S(p) : [0,1] \to \mathbb{R}$; they are the stopping functions.

The discovery phase ends the first time the DCA reaches an estimation round t at which $\sigma_t^B(p_{t-1}^B) \geqslant \sigma_t^S(p_{t-1}^S)$. At the end of the discovery phase, $r^B = p_t^B = p_{t-1}^B$ and $r^S = p_t^S = p_{t-1}^S$ are selected as the buyers and sellers' reserve prices and $q = \min\{D_t, S_t\}$ is chosen as the quantity traded in the allocation phase of the DCA. The stopping functions differ depending on the designer's objective. When the designer's objective is efficiency, the stopping functions $\sigma_t^B(\cdot)$ and $\sigma_t^S(\cdot)$ are the identity functions; when the designer's objective is profit, $\sigma_t^B(\cdot) = \widehat{MR}_t(\cdot)$ and $\sigma_t^S(\cdot) = \widehat{MC}_t(\cdot)$. In the latter case, we will assume that $\widehat{MR}_t(p)$ and $\widehat{MC}_t(p)$ are increasing functions of price.

If $\sigma_t^B(p_{t-1}^B) < \sigma_t^S(p_{t-1}^S)$, then the discovery phase continues; we must distinguish three different cases, depending on estimated excess demand, called the *buyers' clock state*, the *sellers' clock state*, and the *double clock state*. We first describe the conditions that lead to these states, and then explain what happens in each state.

If estimated excess demand is positive, i.e., $\widehat{Z}_t\left(p_{t-1}^B, p_{t-1}^S\right) = \widehat{D}_t(p_{t-1}^B) - \widehat{S}_t(p_{t-1}^S) > 0$, then

[33] As in Ausubel (2004), if we allowed either full or aggregate bid information, then sincere bidding would be an ex post perfect (as opposed to dominant strategy) equilibrium.

[34] We will use t to denote both calendar time, in which case t is any non-negative real number, or rounds, in which case t is a positive integer.

the DCA transitions into a *buyers' clock state* in which the sellers cannot act and each buyer may reduce her demand as the buyers' price increases, with the following target price as an upper bound:

$$p_t^{BT} = \min\left\{\inf\left\{p : \sigma_t^B(p) \geqslant \sigma_t^S(p_{t-1}^S)\right\},\ \sup\left\{p : \widehat{D}_t(p) \geqslant \widehat{S}_t(p_{t-1}^S)\right\}\right\}.$$

If estimated excess demand is negative, i.e., $\widehat{Z}_t\left(p_{t-1}^B, p_{t-1}^S\right) < 0$, then the DCA transitions into a *sellers' clock state* in which the buyers cannot act and each seller may reduce her supply as the sellers' price decreases, with the following target price as a lower bound:

$$p_t^{ST} = \max\left\{\sup\left\{p : \sigma_t^B(p_{t-1}^B) \geqslant \sigma_t^S(p)\right\},\ \inf\left\{p : \widehat{D}_t(p_{t-1}^B) \leqslant \widehat{S}_t(p)\right\}\right\}.$$

If estimated demand and supply are equal, i.e., $\widehat{Z}_t\left(p_{t-1}^B, p_{t-1}^S\right) = 0$, then the DCA enters a *double clock state*, with buyers and sellers both active; the buyers' price increases and the sellers' price decreases with target prices that solve, for all $\epsilon > 0$:

$$\begin{aligned} \widehat{D}_t(p_t^{BT} - \epsilon) &\geqslant \widehat{S}_t(p^{ST} - \epsilon), \qquad & \widehat{D}_t(p_t^{BT} + \epsilon) &\leqslant \widehat{S}_t(p^{ST} + \epsilon), \\ \sigma_t^B(p_t^{BT} - \epsilon) &\leqslant \sigma_t^S(p^{ST} - \epsilon), \qquad & \sigma_t^B(p_t^{BT} + \epsilon) &\geqslant \sigma_t^S(p^{ST} + \epsilon). \end{aligned}$$

To see how the estimation phase works, suppose for simplicity that estimated demand and supply are continuous functions when efficiency is the designer's goal and, in addition, estimated marginal cost and revenue are continuous functions when profit is the goal. Then, in the definitions of the rules for ending the estimation phase and of the target prices, weak inequalities are replaced by equalities and inf and sup are replaced by min and max.

Consider first the case when the designer's objective is efficiency, so that the stopping functions $\sigma_t^B(\cdot)$ and $\sigma_t^S(\cdot)$ are the identity functions. The buyers' target price p_t^{BT} when estimated excess demand is positive is the minimum between the current sellers' clock price p_{t-1}^S and the price at which estimated demand equals estimated supply at p_{t-1}^S. Similarly, the sellers' target price p_t^{ST} when estimated excess demand is negative is the maximum between the current buyers' clock price p_{t-1}^B and the price at which estimated supply equals estimated demand at p_{t-1}^B. When estimated demand equals estimated supply at the current clock prices, the common target price is the price at which estimated demand equals estimated supply. The discovery phase ends in the first round t with initial clock prices satisfying $p_{t-1}^B = p_{t-1}^S$.[35] Denote by r such a common ending price (to be used as the reserve price in the allocation phase). It corresponds to the target price that was reached at the end of round $t-1$; hence no trader has dropped out in round $t-1$ and estimated demand and supply at t are unchanged from $t-1$. Consequently, estimated demand is "as close as possible" to estimated supply.

[35] As the clocks will not move in round t, the final prices will be the same as the initial prices, $p_t^B = p_{t-1}^B$, $p_t^S = p_{t-1}^S$.

We have $\widehat{D}_t(r) \geqslant \widehat{S}_t(r)$ if round $t-1$ ended in a buyers' clock state, $\widehat{D}_t(r) \leqslant \widehat{S}_t(r)$ if it ended in a sellers' clock state, and $\widehat{D}_t(r) = \widehat{S}_t(r)$ if it ended in a double clock state.

Consider now the case when the designer's objective is profit, so that the stopping functions are $\sigma_t^B(\cdot) = \widehat{MR}_t(\cdot)$ and $\sigma_t^S(\cdot) = \widehat{MC}_t(\cdot)$. The buyers' target price p_t^{BT} when estimated excess demand is positive is the minimum between the price at which marginal revenue equates marginal cost evaluated at the current sellers' clock price p_{t-1}^S and the price at which estimated demand equals estimated supply at p_{t-1}^S. Similarly, the sellers' target price p_t^{ST} when estimated excess demand is negative is the maximum between the price at which marginal cost equates marginal revenue at the current buyers' clock price p_{t-1}^B and the price at which estimated supply equals estimated demand at p_{t-1}^B. When estimated demand equals estimated supply at the current clock prices, the target prices p^{BT} and p^{ST} are the prices at which estimated marginal revenue equates estimated marginal cost and estimated demand equals estimated supply. The discovery phase ends in the first round t with initial clock prices $p_{t-1}^B = r^B$ and $p_{t-1}^S = r^S$ satisfying $\widehat{MR}_t(r^B) \geqslant \widehat{MC}_t(r^S)$.[36] In this sense, estimated demand is as close as possible to estimated supply and estimated marginal revenue is as close as possible to estimated marginal procurement cost. If round $t-1$ ended in a buyers' clock state, then either the target price was reached and $\widehat{MR}_{t-1}(r^B) = \widehat{MC}_{t-1}(r^S)$, or a buyer dropped out and $\widehat{MR}_t(r^B) \geqslant \widehat{MC}_t(r^S)$; in both cases, $\widehat{D}_{t-1}(r^B) \geqslant \widehat{S}_{t-1}(r^S)$. Similarly, if round $t-1$ ended in a sellers' clock state, then either the target price was reached and $\widehat{MR}_{t-1}(r^B) = \widehat{MC}_{t-1}(r^S)$, or a seller dropped out and $\widehat{MR}_t(r^B) \geqslant \widehat{MC}_t(r^S)$; in both cases, $\widehat{D}_{t-1}(r^B) \leqslant \widehat{S}_{t-1}(r^S)$. If round $t-1$ ended in a double clock state, then either the target prices were reached and $\widehat{MR}_{t-1}(r^B) = \widehat{MC}_{t-1}(r^S)$, or a trader dropped out and $\widehat{MR}_t(r^B) \geqslant \widehat{MC}_t(r^S)$; in both cases, $\widehat{D}_{t-1}(r^B) = \widehat{S}_{t-1}(r^S)$.

Discovery Phase: The Buyers' Clock State. Suppose that in round t the DCA has transitioned into the buyers' clock state with initial clock prices p_{t-1}^B and p_{t-1}^S and buyers' target price p_t^{BT}. The set of active buyers is $\mathcal{N}\backslash I_t^B$ and the initial demand of any active buyer b is $d_{t-1}^b > 0$. Then:

1. The sellers' clock does not move, the final sellers' clock price of round t is $p_t^S = p_{t-1}^S$.
2. The buyers' clock price increases continuously starting from p_{t-1}^B at such a rate that it reaches the target price p^{BT} in a unit of time.
3. Any active buyer $b \in \mathcal{N}\backslash I_t^B$ may reduce her quantity demanded at any point in time.
4. The buyers' clock stops either when the price reaches its target, or at the first price p_τ^B, $\tau \in [t-1, t]$, at which an active buyer drops out (i.e., $d_\tau^b = 0$ for a $b \in \mathcal{N}\backslash I_t^B$). If several buyers select to drop out at the same time, a priority rule selects one of them as

[36]The discovery phase continues as long as $\widehat{MR}_t(p_{t-1}^B) < \widehat{MC}_t(p_{t-1}^S)$ and would ideally stop when estimated marginal revenue and cost are equal, but following a trader drop out at $t-1$, it might happen that the re-estimation at round t gives marginal revenue above marginal cost.

the dropping-out buyer b for which $d^b_\tau = 0$ while the quantities demanded by the others remain what they were before they selected to drop out.

5. After the buyers' clock has stopped and time t has been reached (i.e., a unit of time has elapsed), the DCA transitions to the next round into an estimation state.

Discovery Phase: The Sellers' Clock State. Suppose in round t the DCA has transitioned into the sellers' clock state with initial clock prices p^B_{t-1} and p^S_{t-1} and sellers' target price p^{ST}_t. The set of active sellers is $\mathcal{M}\backslash I^S_t$ and the initial quantity of any active seller s is $s^s_{t-1} > 0$. Then:

1. The buyers' clock does not move, the final buyers' clock price of round t is $p^B_t = p^B_{t-1}$.
2. The sellers' clock price increases continuously starting from p^S_{t-1} at a rate that will make it reach the target price p^{ST} in a unit of time.
3. Any active seller $s \in \mathcal{M}\backslash I^S_t$ may reduce her quantity supplied at any point in time.
4. The sellers' clock stops either when the price reaches its target, or at the first price p^S_τ, $\tau \in [t-1, t]$ at which an active seller drops out (i.e., $s^s_\tau = 0$ for an $s \in \mathcal{M}\backslash I^S_t$). If several sellers select to drop out at the same time, a priority rule selects one of them as the dropping-out seller s for which $s^s_\tau = 0$ while the quantities supplied by the others remain what they were before they selected to drop out.
5. After the sellers' clock has stopped and time t has been reached, the DCA transitions to the next round into an estimation state.

Discovery Phase: The Double Clock State. Suppose in round t the DCA has transitioned into the double clock state with initial clock prices p^B_{t-1} and p^S_{t-1} and target prices p^{BT}_t and p^{ST}_t. The set of active traders is $(\mathcal{N} \cup \mathcal{M})\backslash I_t$, the initial quantity of an active buyer b is d^b_{t-1} and the initial quantity of an active seller s is $s^s_{t-1} > 0$. Then:

1. The buyers' clock price increases continuously starting from p^B_{t-1} and the sellers' clock price increases continuously starting from p^S_{t-1}; they change at such rates that they reach their target prices in a unit of time absent additional exits.
2. At any point in time any active buyer b may reduce her quantity demanded and any active seller s may reduce her quantity supplied.
3. Both clocks stop together either when the prices simultaneously reach their targets, or at the first prices p^B_τ and p^S_τ, $\tau \in [t-1, t]$, at which an active trader drops out (i.e., $d^b_\tau = 0$ for a $b \in \mathcal{N}\backslash I^B_t$, or $s^s_\tau = 0$ for an $s \in \mathcal{M}\backslash I^S_t$). If several traders select to drop out at the same time, a priority rule selects one of them as the dropping-out trader, b for which $d^b_\tau = 0$ or s for which $s^s_\tau = 0$, while the quantities demanded and supplied by the others remain what they were before they selected to drop out.

4. After the buyers' and sellers' clocks have stopped and time t has been reached, the DCA transitions to the next round into an estimation state.

The discovery phase will end in finite time, as there are a finite number of agents.

The Allocation Phase. Let $t = E$ be the time at which the discovery phase ends (and the allocation phase begins). The aggregate quantity traded in the DCA is $q = \min\{D_E, S_E\}$ and the reserve prices are $r^B = p^B_E$ for buyers and $r^S = p^S_E$ for sellers. At least one side trades all units at their side's reserve price. If $D_E = q$, all buyers obtain the quantities they demand at the reserve price r^B; likewise, if $S_E = q$ all sellers trade the quantities they supply at the sellers' reserve price r^S. If $D_E > q$, then an Ausubel auction determines the prices and quantities acquired by each buyer who is active at E (i.e., is demanding a positive quantity). As in the discovery phase, at any point in time buyer b indicates the quantity d^b_t she is willing to buy at the current price, the buyers' clock price weakly increases continuously (starting at $p^B_E = r^B$), and buyers are subject to a monotone activity rule: $d^b_{t'} \leqslant d^b_t$ for all $b \in \mathcal{N}$ and all t, t' such that $t' > t$. Similarly, if $S_E > q$, a 'reverse' Ausubel auction determines quantities acquired and prices for all sellers who are active at E (those supplying a positive quantity). At any time t seller s indicates the quantity s^s_t she is willing to sell, the seller's clock price weakly decreases continuously (starting at $p^S_E = r^S$), and sellers are subject to a monotone activity rule: $s^s_{t'} \leqslant s^s_t$ for all $s \in \mathcal{M}$ and all t, t' such that $t' > t$.

For completeness, we now describe the buyers' side and the sellers' side Ausubel auction (see Ausubel, 2004). There are times τ when traders "clinch" additional units (i.e., acquire units if buyers, sell units if sellers). Let $x^b_t \geqslant 0$ and $x^s_t \geqslant 0$ be the additional units clinched at time t by buyer b and seller s. As there are only a discrete number of units for sale, the set of clinching times is finite; denote by $\mathcal{T}^B_t = \{\tau < t : \sum_{b \in \mathcal{N}} x^b_\tau > 0\}$ the set of clinching times before t in the buyers' side Ausubel auction, and by $\mathcal{T}^S_t = \{\tau < t : \sum_{s \in \mathcal{M}} x^s_\tau > 0\}$ the set of clincing times before t in the supply-side auction. By definition, $\mathcal{T}^B_E = \mathcal{T}^S_E = \varnothing$ as no units are clinched before the start of the allocation phase. Define the total, or cumulative, number of units clinched by buyer b and seller s at time t or before as: $X^b_t = x^b_t + X^b_{t-}$ and $X^s_t = x^s_t + X^s_{t-}$, where $X^b_{t-} = \sum_{\tau \in \mathcal{T}^B_t} x^b_\tau$ is the total number of units clinched by b before t and $X^s_{t-} = \sum_{\tau \in \mathcal{T}^S_t} x^s_\tau$ is the total number of units clinched by s before t.

Consider the buyers' side Ausubel auction. A buyer b cannot be allocated less units than the ones she has already clinched; thus, denote by $\widetilde{d}^b_t = \max\{d^b_t, X^b_{t-}\}$ the effective demand of buyer b at t. Let $\widetilde{D}_t = \sum_{b \in \mathcal{N}} \widetilde{d}^b_t$ be aggregate effective demand and $\widetilde{D}^{-b}_t = \sum_{b' \in \mathcal{N} \setminus \{b\}} \widetilde{d}^{b'}_t$ be effective demand at t of all buyers except b. The auction ends at the first time $t = L > E$ such that $\widetilde{D}_L \leqslant q$. For all $t \in [E, L)$ the cumulative clinches by buyer b at t are defined as: $X^b_t = \max\left\{0, q - \widetilde{D}^{-b}_t\right\}$. The clinching rule just described also applies at $t = L$ if $\widetilde{D}_L = q$. If, in contrast, $\widetilde{D}_L < q$, it must be that buyers reduced demand by more than one unit at time

L.[37] We then allocate the items not clinched before L according to any rule that satisfies the following two properties: (i) $0 \leqslant x_L^b \leqslant \min_{t<L}\{\widetilde{d}_t^b - X_t^b\}$; (ii) $\sum_{b\in\mathcal{N}} X_L^b = q$. In the auction, each buyer pays a price for each unit equal to the buyers' clock price when that unit is clinched. Hence the total payment by buyer b is $\sum_{\tau\in\mathcal{T}_L^B\cup L} p_\tau^B \cdot x_\tau^b$.

In the sellers' side Ausubel auction, a seller s must sell at least as many units as she has already clinched; denote by $\widetilde{s}_t^s = \max\{s_t^s, X_{t-}^s\}$ the effective supply of seller s at t. Let $\widetilde{S}_t = \sum_{s\in\mathcal{M}} \widetilde{s}_t^s$ be aggregate effective supply and $\widetilde{S}_t^{-s} = \sum_{s'\in\mathcal{M}\setminus\{s\}} \widetilde{s}_t^{s'}$ be effective supply at t of all sellers except s. The auction ends at the first time $t = L > E$ such that $\widetilde{S}_L \leqslant q$. For all $t \in [E, L)$ the cumulative clinches at t are defined as: $X_t^s = \max\left\{0, q - \widetilde{S}_t^{-s}\right\}$. At $t = L$ the clinching rule just described applies if $\widetilde{S}_L = q$, while if $\widetilde{S}_L < q$ we allocate the items not clinched before L according to any clinching rule that satisfies the two properties: (i) $0 \leqslant x_L^s \leqslant \min_{t<L}\{\widetilde{s}_t^s - X_t^s\}$; (ii) $\sum_{s\in\mathcal{M}} X_L^s = q$. In the auction, each seller is paid a price for each unit sold equal to the sellers' clock price when that unit is clinched; the total payment to seller s is $\sum_{\tau\in\mathcal{T}_L^S\cup L} p_\tau^S \cdot x_\tau^s$.

In line with our assumption that no bid information is revealed, agents only learn the number of units they clinched at time L.

Before proving that sincere bidding is a dominant strategy equilibrium in the DCA, we would like to remark that an additional property of the DCA is a form of constrained efficiency, where the constraint is the total quantity traded.[38] Given the quantity to be traded, the allocation phase guarantees that the trades completed are the most valuable ones – those that maximize consumption value and minimize production cost.

Theorem 2. (Sincere Bidding is a dominant strategy equilibrium in the DCA.) *Sincere bidding by each agent is a dominant strategy equilibrium in the double clock auction with estimation-based tâtonnement. The DCA is also feasible, deficit free and ex post individually rational.*

Proof. By construction, the DCA is feasible as the quantity traded is determined by the short side of the market at the reserve prices, and it is deficit free since the minimum price paid by buyers (the reserve price r^B) is at least as high as the maximum price paid to sellers (the reserve price r^S). Ex post individual rationality holds since each trader may guarantee herself the outside option payoff by dropping out of the bidding. It remains to show that

[37] This is because $\widetilde{D}_t > q$ for all $t < L$ and the buyers' clock price moves continuously. Under sincere bidding, this implies that there must be a tie, with at least two marginal values equal to p_L^B.

[38] Note that sincere bidding by each agent being a dominant strategy equilibrium does not imply that for a given buyer b or seller s sincere bidding weakly dominates all other strategies. Without making more detailed assumptions about the estimation procedure used by the designer, we cannot rule out that sincere bidding by buyer b or seller s always yields the same outcome as bidding ϵ more or less than the marginal value, or cost, on the k-th unit. This could happen when the estimation procedure never leads to a price for the k-th unit in the gap between marginal value, or cost, and the bid.

sincere bidding is a dominant strategy equilibrium. Because of the symmetry of buyers and sellers, we will just show that bidding sincerely for a bidder b is a best reply irrespectively of the strategies of all other traders. The proof that irrespectively of the strategies of all other traders bidding sincerely is also a best response for any seller s is analogous.

Take the bidding strategies of all except buyer b as given. Consider bidder b bidding $\boldsymbol{\beta}^b = (\beta_1^b, ..., \beta_{k_B}^b)$, where β_k^b is the buyers' clock price at which bidder b's demand drops from k to $k-1$. Let r_*^B and r_*^S be the reserve prices resulting and q_*^B, q_*^S the quantities demanded and supplied at the reserve prices. Let $\min\{q_*^B, q_*^S\} = q_*$ be the total quantity traded and q_*^b be the quantity traded by b. Note that in general the vector of prices at which buyer b could obtain (clinch) each unit only depends on the bids of all other buyers and on the total quantity traded q_*, which in turn depends on all bids including b's. Let $\beta_{(h)}^{-b}$ be the h-th highest bid by all buyers except b at the given strategies; that is, if the buyers' clock price reaches price $\beta_{(h)}^{-b}$ then aggregate demand by all buyers except b drops from h to $h-1$. Then the price that buyer b must pay in the DCA in order to acquire the k-th unit is $p_{k*}^b = \max\{r_*^B, \beta_{(q_*+1-k)}^{-b},\}$.

Consider now bidder b bidding sincerely instead – that is, bidding $\boldsymbol{v}^b = (v_1^b, ..., v_{k_B}^b)$ – while all other traders use the same strategies; let the notation for reserve prices r^B, r^S, quantities traded q^B, q^S, q, q^b and b's price for the k-th unit p_k^b differ by dropping the subscript $_*$. For ease of reference, we will refer to buyer b bidding $\boldsymbol{\beta}^b$ as non-sincere bidding. There are four cases. We will show that in each of them bidder b's payoff is at least as high under sincere bidding as when bidding $\boldsymbol{\beta}^b$.

Case 1. If $q_*^b = 0$, then bidding $\boldsymbol{v}^b$ instead of $\boldsymbol{\beta}^b$ cannot reduce bidder b's payoff, since under sincere bidding bidder b never pays for a unit more than her marginal value.

Case 2. If $q_*^b > 0$ and $q_* = q$, then the total quantity traded is the same under sincere and non-sincere bidding by b. There are two subcases. In subcase 2.1, $q^b > 0$. Consequently, under the two bidding strategies of bidder b estimation and hence the reserve prices cannot depend on b's bid and must be the same. In particular, $r^B = r_*^B$ and since $q = q_*$ under the two strategies bidder b faces the same clinching price for each unit she may acquire. She obtains all the units k that have a marginal value v_k^b higher than the clinching price p_{k*}^b under sincere bidding, and hence buyer b cannot be worse off than under non-sincere bidding. In subcase 2.2, $q^b = 0$; under sincere bidding buyer b drops out of the DCA. There are two possible scenarios. If bidder b drops out in the discovery phase, this must happen at a price below r_*^B. (Recall that buyer b cannot influence estimation before dropping out from the discovery phase and if r_*^B were reached under sincere bidding before b drops out, then the discovery phase would stop.) If bidder b drops out in the allocation phase of the DCA, then she drops out while facing the same unit prices as under non-sincere bidding. Consequently, in both scenarios under sincere bidding buyer b's payoff is zero, while under non-sincere bidding she acquires a positive quantity of the good at unit prices weakly above her marginal values.

Case 3. If $q_*^b > 0$ and $q = q_* + \delta$, with δ a positive integer, then under sincere bidding by buyer b the total quantity traded increases by an amount equal to δ. There are two subcases. In subcase 3.1, $q^b > 0$. Since b does not drop out, estimation under sincere and non-sincere bidding and hence the reserve prices must be the same. Consequently, under sincere bidding the unit prices at which buyer b may clinch unit k are weakly lower for each k: $p_k^b = \max\{r_*^B, \beta_{(q_*+\delta+1-k)}^{-b}\} \leqslant \max\{r_*^B, \beta_{(q_*+1-k)}^{-b} = p_{k*}^b\}$. Since under sincere bidding buyer b acquires all units that have a clinching price below the marginal value, it follows that buyer b's payoff is at least as high under sincere as under non-sincere bidding. In subcase 3.2, $q^b = 0$. If b drops out in the discovery phase, then by the same argument as in the first scenario of subcase 2.2 this must happen at a price below r_*^B. If bidder b drops out in the allocation phase of the DCA, then she drops out while facing weakly lower unit prices than under non-sincere bidding. Consequently, under sincere bidding buyer b's payoff is zero, while under non-sincere bidding she acquires a positive quantity of the good at unit prices weakly above her marginal values.

Case 4. If $q_*^b > 0$ and $q = q_* - \delta$, with δ a positive integer, then under sincere bidding by buyer b the total quantity traded decreases by δ. There are two subcases. In subcase 4.1, $q^b > 0$ and estimation under sincere and non-sincere bidding and hence the reserve prices must be the same. Consequently, under sincere bidding the quantity demanded by buyer b at r_*^B must be smaller and buyers must be on the short side of the market. Hence, under sincere bidding buyer b obtains all the units that have a marginal value above r_*^B at price r_*^B. Since under non-sincere bidding the clinching price for each unit is at least r_*^B, buyer b's payoff is at least as high under sincere as under non-sincere bidding. In subcase 4.2, $q^b = 0$. This cannot happen with bidder b dropping out in the allocation phase under sincere bidding, because as in subcase 4.1 buyers would be on the short side of the market and buyer b would be allocated the positive quantity demanded at r_*^B. If b drops out in the discovery phase, then by the same argument as in the first scenario of subcase 2.2 this must happen at a price below r_*^B. Consequently, under sincere bidding buyer b's payoff is zero, while under non-sincere bidding she acquires a positive quantity of the good at unit prices weakly above her marginal values. □

6 The Two-Sided VCG Auction with Reserve Prices

By the revelation principle, there is a direct mechanism that is strategically equivalent to our DCA. In this section we show that such a mechanism is a two-sided VCG auction augmented with endogenously determined reserve prices.

A *two-sided VCG auction with endogenous reserve prices* is a direct mechanism in which all traders simultaneously report their demand and supply functions (i.e., types) to the designer and that:

1. Sets the buyers and sellers' reserve prices $r^B(\boldsymbol{\theta})$ and $r^S(\boldsymbol{\theta})$, with $r^B(\boldsymbol{\theta}) \geqslant r^S(\boldsymbol{\theta})$ and the total quantity traded $q(\boldsymbol{\theta}) = q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta})$ as specified in the discovery phase of the DCA under sincere bidding.[39]
2. Determines that all traders that would drop out in the discovery phase of the DCA under sincere bidding are inactive and trade zero units. All other traders are deemed active.[40]
3. Selects individualized price vectors $p^b(\boldsymbol{\theta}) = \left(0, p_1^b(\boldsymbol{\theta}), ..., p_k^b(\boldsymbol{\theta}), ...\right)$ for active buyer b and $p^s(\boldsymbol{\theta}) = (0, p_1^s(\boldsymbol{\theta}), ..., p_k^s(\boldsymbol{\theta}), ...)$ for active seller s with, for $k > 0$:
$$p_k^b(\boldsymbol{\theta}) = \max\left\{r^B(\boldsymbol{\theta}), \boldsymbol{v}^{-b}_{(q(\boldsymbol{\theta})+1-k)}\right\} \quad \text{and} \quad p_k^s(\boldsymbol{\theta}) = \min\left\{r^S(\boldsymbol{\theta}), \boldsymbol{c}^{-s}_{[q(\boldsymbol{\theta})+1-k]}\right\}.$$
4. Determines that each active agent trades a quantity that maximizes her payoff at the reported type, given the personalized prices; that is,
$$q^b(\boldsymbol{\theta}) \in \arg\max_q \sum_{k=0}^{q} \left(v_k^b - p_k^b(\boldsymbol{\theta})\right) \quad \text{and} \quad q^s(\boldsymbol{\theta}) \in \arg\max_q \sum_{k=0}^{q} \left(p_k^s(\boldsymbol{\theta}) - c_k^s\right).$$

It is an immediate corollary of Theorem 2 that the two-sided VCG auction with endogenous reserve prices is deficit free, satisfies feasibility, ex post individual rationality and dominant strategy incentive compatibility (i.e., truthfully reporting one's own type is a dominant strategy equilibrium). However, it does not satisfy privacy preservation, as traders must report all their marginal values and costs to the designer.

Corollary 1. *In the two-sided VCG auction with endogenous reserve prices, truthful reporting by each agent is a dominant strategy equilibrium. The two-sided VCG auction with endogenous reserve prices is also feasible, deficit free and ex post individually rational.*

7 Quantity Constraints

In many real word settings, groups of bidders may be constrained in the quantities they are allowed to trade, for example because of technological constraints, or perceived needs to limit their market power within the allocation mechanism or in the ensuing product market. As a case in point, for the upcoming FCC "incentive auction" there was a debate about whether AT&T and Verizon should be excluded from participation or restricted in the quantities they could acquire.[41]

[39] To do so, after collecting the reports the designer may "run" the discovery phase of the DCA as if sincere bidding was the strategy used by each trader.

[40] Note that active traders might still trade zero units.

[41] For a rebuttal of arguments in favour of exclusion, see Marx (2013).

In this section, we show how our DCA can be adjusted to deal with quantity constraints. Let $\{\mathcal{B}_i\}_{i=1}^{n_B}$ be a partition of the set of buyers $\mathcal{N}$ into n_B disjoint groups $\mathcal{B}_i$ and $\{\mathcal{S}_j\}_{j=1}^{m_S}$ be a partition of the set of sellers $\mathcal{M}$ into m_S disjoint groups $\mathcal{S}_j$. Let $|\mathcal{B}_i|$ be the number of buyers in group $\mathcal{B}_i$ and $|\mathcal{S}_j|$ be the number of sellers in group $\mathcal{S}_j$.

The quantity constraint imposed on buyers in group $\mathcal{B}_i$ and sellers in group $\mathcal{S}_j$ are $q_{\mathcal{B}_i} \leqslant |\mathcal{B}_i| k_B$ and $q_{\mathcal{S}_j} \leqslant |\mathcal{S}_j| k_S$, where $q_{\mathcal{B}_i}$ is the maximum total quantity that can be acquired by group $\mathcal{B}_i$ and $q_{\mathcal{S}_j}$ is the maximum total quantity that can be sold by group $\mathcal{S}_j$. Note that the constraint on a group has bite only when the inequality is strict; when the inequality holds as an equality the group is unconstrained.[42] These constraints can be accommodated by introducing minor modifications to our DCA.

The only formal change in the discovery phase must be made at the end.[43] Recall that the discovery phase ends at the first estimation round $t = E$ when $\sigma_t^B(p_{t-1}^B) \geqslant \sigma_t^S(p_{t-1}^S)$ yielding $r^B = p_E^B = p_{t-1}^B$ as the buyers' reserve price and $r^S = p_E^S = p_{t-1}^S$ as the sellers' reserve price. In order to compute the total quantity traded q we need to take into account the quantity constraints. We do so by first defining constrained demand by group $\mathcal{B}_i$ at time t, $\overline{D}_t^{\mathcal{B}_i}$, constrained market demand at t, $\overline{D}_t$, constrained supply by group $\mathcal{S}_j$ at t, $\overline{S}_t^{\mathcal{S}_j}$, and constrained supply at t, $\overline{S}_t$:[44]

$$\overline{D}_t^{\mathcal{B}_i} = \min\left\{\sum_{b\in\mathcal{B}_i} \widetilde{d}_t^b, q_{\mathcal{B}_i}\right\}, \qquad \overline{D}_t = \sum_{i=1}^{n_B} \overline{D}_t^{\mathcal{B}_i},$$

$$\overline{S}_t^{\mathcal{S}_j} = \min\left\{\sum_{s\in\mathcal{S}_j} \widetilde{s}_t^s, q_{\mathcal{S}_j}\right\}, \qquad \overline{S}_t = \sum_{j=1}^{m_S} \overline{S}_t^{\mathcal{S}_j}.$$

We then specify that the quantity traded is $q = \min\{\overline{D}_E, \overline{S}_E\}$. We say that buyers (sellers) are on the short side of the market if $\overline{D}_E = q$ ($\overline{S}_E = q$); we say that buyers (sellers) are on the long side of the market if $\overline{D}_E > q$ ($\overline{S}_E > q$).[45] We also say that the quantity constraint on group $\mathcal{B}_i$ binds if $\sum_{b\in\mathcal{B}_i} \widetilde{d}_E^b > q_{\mathcal{B}_i}$; analogously, the quantity constraint on group $\mathcal{S}_j$ binds if $\sum_{s\in\mathcal{S}_j} \widetilde{s}_E^s > q_{\mathcal{S}_j}$.

The allocation phase of the DCA must be modified as follows. If no quantity constraints bind on the short side, then all agents on the short side trade at the short side reserve price. If, on the other hand, one or more group specific constraints on the short side bind, separate

[42]Also note that individual quantity constraints could be easily added to the group constraints. If buyer $b \in \mathcal{B}_i$ is also not allowed to purchase more than k^b units and seller $s \in \mathcal{S}_j$ is also not allowed to sell more than k^s units, it suffices to restrict the demand by b and supply by s at the beginning of the DCA to be at most k^b and k^s, respectively.

[43]Of course, the market maker may want to account for the quantity constraints in order to better estimate demand, supply, marginal revenue and marginal cost.

[44]Recall that in the discovery phase no units are clinched and hence $\widetilde{d}_t^b = d_t^b$ and $\widetilde{s}_t^s = s_t^s$.

[45]Our terminology implies that buyers and sellers could both be on the short side of the market, but they cannot both be on the long side.

Ausubel auctions are run, starting at the short side reserve price, one Ausubel auction for each group with a binding constraint to allocate the maximum total quantity to the group's members.[46] Now consider the long side. If no group specific quantity constraint binds, then a single Ausubel auction is run to allocate the q units efficiently with the long side reserve price as the starting price. If some group specific constraints bind at the reserve, then a quantity constrained Ausubel auction is run.

We first define the buyers' side, quantity constrained, Ausubel auction. As in the unconstrained version, each buyer pays a price for each unit equal to the buyers' clock price when that unit is individually clinched. However, when a group specific quantity constraint binds, units are first clinched by the group and then individually clinched by its members. Consider buyer $b \in \mathcal{B}_i$; let $\overline{D}_t^{\mathcal{B}_i \setminus \{b\}} = \min \left\{ \sum_{b_i \in \mathcal{B}_i \setminus \{b\}} \widetilde{d}_t^{b_i}, q_{\mathcal{B}_i} \right\}$ be the constrained demand at t of buyers in group $\mathcal{B}_i$ except b, and let $\overline{D}_t^{-b} = \overline{D}_t^{\mathcal{B}_i \setminus \{b\}} + \sum_{i' \neq i} \overline{D}_t^{\mathcal{B}_{i'}}$ be the constrained demand at t of all buyers except b. Let $L > E$ be the first time t such that $\overline{D}_t \leqslant q$. Define the cumulative clinches by buyer b at $t \in [E, L)$ as:

$$X_t^b = \max \left\{ 0, \min \left\{ q_{\mathcal{B}_i} - \overline{D}_t^{\mathcal{B}_i \setminus \{b\}}, q - \overline{D}_t^{-b} \right\} \right\}. \tag{2}$$

At $t = L$ we distinguish between two cases. First, if $\overline{D}_L = q$, then the clinching rule (2) applies. Note that it need not be the case that $\sum_{b \in \mathcal{N}} X_L^b = q$. It could be $\sum_{b \in \mathcal{N}} X_L^b < q$, because there could be a group $\mathcal{B}_i$ with unconstrained aggregate demand exceeding the maximum quantity $q_{\mathcal{B}_i}$; that is, it could be $\sum_{b \in \mathcal{B}_i} \widetilde{d}_L^b > q_{\mathcal{B}_i}$ and, as a consequence, $\overline{D}_L^{\mathcal{B}_i} = q_{\mathcal{B}_i} > \sum_{b \in \mathcal{B}_i} X_L^b$. In such a case, we say that group $\mathcal{B}_i$ clinches $x_L^{\mathcal{B}_i} = q_{\mathcal{B}_i} - \sum_{b \in \mathcal{B}_i} X_L^b$ units at $t = L$ and we run an Ausubel auction with starting price p_L^B among the agents in $\mathcal{B}_i$ to determine the individual clinches of the $x_L^{\mathcal{B}_i}$ units. Second, if $\overline{D}_L < q$, then we allocate the items not clinched before L either to individual buyers or to groups, according to any rule that satisfies the following properties: (i) $0 \leqslant x_L^b \leqslant \min_{t<L} \{ \widetilde{d}_t^b - X_t^b \}$; (ii) $\sum_{b \in \mathcal{B}_i} X_L^b \leqslant q_{\mathcal{B}_i}$; (iii) if $\overline{D}_L^{\mathcal{B}_i} = q_{\mathcal{B}_i}$, then $x_L^{\mathcal{B}_i} = q_{\mathcal{B}_i} - \sum_{b \in \mathcal{B}_i} X_L^b$; (iv) $\sum_{b \in \mathcal{N}} X_L^b + \sum_{i=1}^{n_B} x_L^{\mathcal{B}_i} = q$. For each group $\mathcal{B}_i$ that clinches $x_L^{\mathcal{B}_i} > 0$ units at $t = L$, we run an Ausubel auction with starting price p_L^B to assign those units to individual members of the group.

The sellers' side, quantity constrained, Ausubel auction is defined analogously, with each seller paid a price for each unit sold equal to the sellers' clock price when that unit is individually clinched. For seller $s \in \mathcal{S}_j$, let $\overline{S}_t^{\mathcal{S}_j \setminus \{s\}} = \min \left\{ \sum_{s_j \in \mathcal{S}_j \setminus \{s\}} \widetilde{s}_t^{s_j}, q_{\mathcal{S}_j} \right\}$ and $\overline{S}_t^{-s} = \overline{S}_t^{\mathcal{S}_j \setminus \{s\}} + \sum_{j' \neq j} \overline{S}_t^{\mathcal{S}_{j'}}$. Let $L > E$ be the first t with $\overline{S}_t \leqslant q$. The cumulative clinches by s at $t \in [E, L)$ are:

$$X_t^s = \max \left\{ 0, \min \left\{ q_{\mathcal{S}_j} - \overline{S}_t^{\mathcal{S}_j \setminus \{s\}}, q - \overline{S}_t^{-s} \right\} \right\}. \tag{3}$$

[46] Note that running these separate auctions is equivalent to running the single, quantity constrained, Ausubel auction defined in the next paragraph.

If $\overline{S}_L = q$, then rule (3) also applies at $t = L$; if, in addition, $\overline{S}_L^{\mathcal{S}_j} = q_{\mathcal{S}_j} > \sum_{s\in\mathcal{S}_j} X_L^s$, then at $t = L$ group $\mathcal{S}_j$ clinches $x_L^{\mathcal{S}_j} = q_{\mathcal{S}_j} - \sum_{s\in\mathcal{S}_j} X_L^s$ units and an Ausubel auction with starting price p_L^S among the agents in $\mathcal{S}_j$ is run for the $x_L^{\mathcal{S}_j}$ units. If $\overline{S}_L < q$, then we allocate the items not clinched before L according to any rule that satisfies the following properties: (i) $0 \leqslant x_L^s \leqslant \min_{t<L}\{\tilde{s}_t^s - X_t^s\}$; (ii) $\sum_{s\in\mathcal{S}_j} X_L^s \leqslant q_{\mathcal{S}_j}$; (iii) if $\overline{S}_L^{\mathcal{S}_j} = q_{\mathcal{S}_j}$, then $x_L^{\mathcal{S}_j} = q_{\mathcal{S}_j} - \sum_{s\in\mathcal{S}_j} X_L^s$; (iv) $\sum_{s\in\mathcal{M}} X_L^s + \sum_{j=1}^{m_S} x_L^{\mathcal{S}_j} = q$. For each group $\mathcal{S}_j$ that clinches $x_L^{\mathcal{S}_j} > 0$ units at $t = L$, we run an Ausubel auction with starting price p_L^S for those units among the group members.

Call the double clock auction just described the *constraint adjusted* DCA. The arguments in the proof of Theorem 2 can be simply extended to establish the following corollary.

Corollary 2. *Sincere bidding by each agent is a dominant strategy equilibrium in the constraint adjusted double clock auction. The constraint adjusted DCA is also feasible, deficit free and ex post individually rational.*

The Example Revisited. To illustrate how the constraint adjusted DCA works, we briefly return to the example of Section 4. Consider the efficiency targeting DCA and suppose that buyers 1 and 2 belong to a first buyers' group with a quantity constraint of 4; buyers 3 and 4 belongs to a second group with a quantity constraint of 3; all other buyers belong to a third group with no constraint (equivalently, a constraint of 12 units). On the sellers' side, sellers 1, 2 and 3 belong to the first sellers' group with a quantity constraint of 3; sellers 4, 5 and 6 belong to the second sellers' group with a constraint of 2 units; all other sellers belong to the third group with a slack constraint.

Observe that if the estimation procedure in the discovery phase is the same linear regression procedure used when there are no quantity constraints, then the discovery phase ends at the same time E and selects the same common reserve price $r^B = r^S = 51.20$. Aggregate constrained demand and supply are $\overline{D}_E = 8$ and $\overline{S}_E = 7$; hence the quantity traded is 7.[47] Sellers are on the short side of the market. The quantity constraint for sellers in the third group is slack and hence sellers 7 and 8 sell one unit each at the reserve price of 51.20. The quantity constraint for sellers in the second group binds; the outcome of the Ausubel auction in this group is that sellers 4 and 5 sell one unit each at the price of 38. The quantity constraint also binds for sellers in the first group and the Ausubel auction in this group starts with seller 1 clinching a sale at the reserve price of 51.20 and ends with sellers 2 and 3 clinching a sale each at the price of 33.

The quantity constrained Ausubel auction for the long side (i.e., buyers) starts at the reserve price of 51.20 with buyer 1 clinching one unit. It then continues and when the buyers' clock price reaches $p_t^B = 54$, buyer 3 clinches one unit; when the clock reaches $p_t^B = 58$ buyer 1 clinches one additional unit. Finally, when the clock reaches the price of 62, buyer 5 drops

[47]The constrained efficient quantity is also 7.

out. At this time, $t = L$, constrained demand equals the quantity available $q = 7$. Indeed, unconstrained demand is also 7 and thus buyers 1 and 3 clinch one additional unit, and buyers 2 and 4 clinch their first unit.

8 Asymptotic Properties of the DCA

To understand the asymptotic properties of the DCA, it is useful to think of the discovery phase as a statistical inference problem and to endow the designer with a model of the random process generating traders' valuations. Rather than looking for the most general model, we will focus on a framework that makes it simple to see the nature of the conditions that guarantee asymptotic efficiency in the efficiency targeting version and asymptotic maximum profit extraction in the profit targeting version.[48]

To that end, we now assume ex ante symmetry of buyers and sellers. A marginal value schedule for a buyer is an element of the set $\mathcal{V} = \{\boldsymbol{v}^b \in [0,1]^{k_B} : v_k^b \geqslant v_{k+1}^b\}$ and a marginal cost schedule for a seller is an element of the set $\mathcal{C} = \{\boldsymbol{c}^s \in [0,1]^{k_S} : c_k^s \leqslant c_{k+1}^s\}$. Consider two families of probability measures on the sigma fields generated by the bounded rectangles in $\mathcal{V}$ and $\mathcal{C}$: the family $\mathcal{F} = \{\mu^{\omega^B};\, \omega^B \in \Omega^B\}$ for buyers and the family $\mathcal{G} = \{\mu^{\omega^S};\, \omega^S \in \Omega^S\}$ for sellers. For buyers and $\boldsymbol{y}^B \in \mathcal{V}$, we denote by $F(\boldsymbol{y}^B;\omega^B) = \mu^{\omega^B}[\boldsymbol{v}^b \in \mathcal{V} : v_k^b \leqslant y_k^B,\, k = 1,...,k_B]$ the distribution function associated with μ^{ω^B}, and by $F_k(y_k^B;\omega^B) = \mu^{\omega^B}[\boldsymbol{v}^b \in \mathcal{V} : v_k^b \leqslant y_k^B]$ the distribution function of the k-th marginal value. For sellers and $\boldsymbol{y}^S \in \mathcal{C}$, we define the reliability function $R(\boldsymbol{y}^S;\omega^S) = \mu^{\omega^S}[\boldsymbol{c}^s \in \mathcal{C} : c_k^s \geqslant y_k^S,\, k = 1,...,k_S]$ associated with μ^{ω^S}, and the reliability function of the k-th marginal cost $R_k(y_k^S;\omega^S) = \mu^{\omega^S}[\boldsymbol{c}^s \in \mathcal{C} : c_k^s \geqslant y_k^S]$.

We assume that the vectors of marginal values of each buyer are independently drawn from one of the measures μ^{ω^B}; similarly, the vectors of marginal costs are independently drawn from one of the measures μ^{ω^S} for each seller. Thus, each element ω^B of the index set Ω^B determines one of the measures from which each buyer marginal values could be independently drawn; similarly, each element ω^S of the set Ω^S determines one of the measures from which each seller marginal costs could be independently drawn. We will assume that Ω^B and Ω^S are compact metric spaces and that $F(\boldsymbol{y}^B;\omega^B)$ and $R(\boldsymbol{y}^S;\omega^S)$ are continuous functions of ω^B and ω^S, respectively.[49]

To study convergence as the number of traders grows large, we will let the number of buyers be nN and the number of sellers be nM and study the limit as $n \to \infty$. To compute market demand and supply at price p we need the following empirical distributions and reliabilities,

[48] In both versions convergence takes place at rate $1/n$ as $n \to \infty$ and the number of buyers and sellers is proportional to n.

[49] These assumptions are only used to make sure that the estimators in (8) and (9) are well defined. They could be replaced by any assumption which guarantees the existence of minimizers in (8) and (9); e.g., the assumption that Ω^B and Ω^S include all simple distribution functions (i.e., all functions that take a countable number of values).

for all $k = 1, ..., k_B$ for buyers and all $k = 1, ..., k_S$ for sellers:

$$F_k^n(p) = \frac{1}{nN}\sum_{b=1}^{nN} \mathbf{1}(v_k^b \leqslant p) \quad \text{and} \quad R_k^n(p) = \frac{1}{nM}\sum_{s=1}^{nM} \mathbf{1}(c_k^s \geqslant p),$$

where $\mathbf{1}(\cdot)$ is the indicator function. Total demand and supply at price p are given by:

$$D^n(p) = nN\sum_{k=1}^{k_B}\left[1 - F_k^n(p)\right] \quad \text{and} \quad S^n(p) = nM\sum_{k=1}^{k_S}\left[1 - R_k^n(p)\right].$$

Since the empirical distributions and reliabilities converge to the true distributions and reliabilities (see the proof of Theorem 3 in the appendix for details), when ω^B and ω^S are the true parameters and n grows large per-capita demand $d^n(p) = \frac{D^n(p)}{nN}$ and supply $s^n(p) = \frac{S^n(p)}{nM}$ are approximately equal to

$$d(p; w^B) = \sum_{k=1}^{k_B}\left[1 - F_k(p; \omega^B)\right] \quad \text{and} \quad s(p; \omega^S) = \sum_{k=1}^{k_S}\left[1 - R_k(p; \omega^S)\right].$$

The following assumption guarantees that with a large number of traders per capita demand and supply are strictly monotone functions.

Assumption 1. (*Monotonicity of Demand and Supply*)
For all $p \in (0, 1)$, all $\epsilon > 0$, all $\omega^B \in \Omega^B$ and all $\omega^S \in \Omega^S$, there exists $\eta > 0$ such that:

$$\sum_{k=1}^{k_B}\left[F_k(p; \omega^B) - F_k(p - \epsilon; \omega^B)\right] > \eta\epsilon \tag{4}$$

$$\sum_{k=1}^{k_S}\left[R_k(p - \epsilon; \omega^S) - R_k(p; \omega^S)\right] > \eta\epsilon \tag{5}$$

A sufficient condition for Assumption 1 to hold is that the measures μ^{ω^B} and μ^{ω^S} are absolutely continuous with respect to Lebesgue measure and their Radon-Nikodym derivatives (densities) are bounded away from zero.

The DCA does not allow the designer to use all the available information to estimate market demand and supply at price p. In computing target prices, when the buyers' clock price in the DCA is p, the designer can only use the marginal value schedules of all the buyers that have dropped out at prices not higher than p. For all $\boldsymbol{y}^B \in \mathcal{V}$ with $y_1^B \leqslant p$, the designer can then compute the following empirical distribution:

$$F^n(\boldsymbol{y}^B; p) = \frac{1}{nN}\sum_{b=1}^{nN} \mathbf{1}(\boldsymbol{v}^b \leqslant \boldsymbol{y}^B) \tag{6}$$

Similarly, when the sellers' clock price is p, the designer can only use the marginal cost schedule of all the sellers that have dropped out at prices not lower than p. For all $\boldsymbol{y}^S \in \mathcal{C}$

with $y_1^S \geqslant p$, the designer can then compute the following empirical reliability:

$$R^n(\boldsymbol{y}^S; p) = \frac{1}{nM} \sum_{s=1}^{nM} \mathbf{1}(\boldsymbol{c}^s \geqslant \boldsymbol{y}^S) \tag{7}$$

Using the empirical distribution for marginal values below p and the empirical reliability for marginal costs above p the designer may compute the following minimum distance parameter estimates:

$$\omega^B(p) \in \arg\min_{\omega^B \in \Omega^B} \sup_{\{\boldsymbol{y}^B \in \mathcal{V}: y_1^B \leqslant p\}} \left| F^n(\boldsymbol{y}^B; p) - F(\boldsymbol{y}^B; \omega^B) \right| \tag{8}$$

$$\omega^S(p) \in \arg\min_{\omega^S \in \Omega^S} \sup_{\{\boldsymbol{y}^S \in \mathcal{C}: y_1^S \geqslant p\}} \left| R^n(\boldsymbol{y}^S; p) - R(\boldsymbol{y}^S; \omega^S) \right| \tag{9}$$

To prove convergence of the DCA we need an additional assumption. Given two admissible measures for buyers, consider the variation in the value they attach to the rectangle $\mathcal{V}_k(p) = [\boldsymbol{y}^B \in \mathcal{V} : y_k^B \leqslant p]$ and the variation in the value they attach to the rectangle $\mathcal{V}_1(p) = [\boldsymbol{y}^B \in \mathcal{V} : y_1^B \leqslant p]$. We assume there is a uniform bound on the ratio of these variations. Similarly, given two admissible measures for sellers, we assume there is a uniform bound on the ratio of the difference in the value they attach to the rectangle $\mathcal{C}_k(p) = [\boldsymbol{y}^S \in \mathcal{C} : y_k^S \geqslant p]$ and the difference in the value they attach to the rectangle $\mathcal{C}_1(p) = [\boldsymbol{y}^S \in \mathcal{C} : y_1^S \geqslant p]$.

Assumption 2. (*Bounded Marginal-Joint Variation Ratio*)
For all $p \in (0,1)$ there exists $\kappa > 0$ s.t.:

$$\left| F_k(p; \omega_0^B) - F_k(p; \omega_1^B) \right| \leqslant \kappa \cdot \sup_{\{\boldsymbol{y}^B \in \mathcal{V}: y_1^B \leqslant p\}} \left| F(\boldsymbol{y}^B; \omega_0^B) - F(\boldsymbol{y}^B; \omega_1^B) \right| \tag{10}$$

for all $\omega_0^B, \omega_1^B \in \Omega^B$ and all $k = 1, ..., k_B$, and

$$\left| R_k(p; \omega_0^S) - R_k(p; \omega_1^S) \right| \leqslant \kappa \cdot \sup_{\{\boldsymbol{y}^S \in \mathcal{C}: y_1^S \geqslant p\}} \left| R(\boldsymbol{y}^S; \omega_0^S) - R(\boldsymbol{y}^S; \omega_1^S) \right| \tag{11}$$

for all $\omega_0^S, \omega_1^S \in \Omega^S$ and all $k = 1, ..., k_S$.

Intuitively, with a large number of traders convergence to efficiency, or maximum profit extraction, obtains if the reserve prices for buyers and sellers converges to the competitive price, or to the prices at which marginal revenue equals marginal procurement cost (and demand equals supply). In the DCA, as the buyers and sellers' clocks move, the designer will be able to make accurate estimates of the measures of the sets $\mathcal{V}_1(p)$ and $\mathcal{C}_1(p)$, as she can only use information from buyers and sellers who are inactive at price p. To estimate demand, supply, marginal revenue and marginal procurement cost, however, the designer needs accurate estimates of the sets $\mathcal{V}_k(p)$ and $\mathcal{C}_k(p)$ for all k. Assuming that there is a uniform bound to the marginal-to-joint variation ratio turns out to be sufficient to guarantee that accurate estimates can be made as the number of traders grows large.

We now present several examples; we begin with one under which Assumption 2 fails.

Example 1. *Suppose Ω^B is the space of continuous functions from $[0,1]$ to $[0,1]$. Let $k_B = 2$ and suppose the distribution of the first marginal value is uniform: $F_1\left(v_1;\omega^B\right) = v_1$ for all ω^B, while the distribution of the second marginal value conditional on the first marginal value being v_1 is $F_2\left(v_2|v_1;\omega^B\right) = \left(\frac{v_2}{v_1}\right)^{1+\omega^B(v_1)}$ for $v_2 \leqslant v_1$.*

To see that Assumption 2 fails, take ω_0^B and ω_1^B such that $\omega_0^B(v) = \omega_1^B(v)$ for all $v \leqslant p$ and $\omega_0^B(v) < \omega_1^B(v)$ for all $v > p$ for some $p \in (0,1)$. For $\{v_1, v_2 : v_1 \leqslant p\}$, we have

$$F\left(v_1,v_2;\omega_0^B\right) = \int_0^{v_1}\int_0^{\min\{x,v_2\}} d\left(\frac{y}{x}\right)^{1+\omega_0^B(x)} dx = F\left(v_1,v_2;\omega_1^B\right) = \int_0^{v_1}\int_0^{\min\{x,v_2\}} d\left(\frac{y}{x}\right)^{1+\omega_1^B(x)} dx$$

$$F_2\left(v_2;\omega_0^B\right) = \int_0^{1}\int_0^{\min\{x,v_2\}} d\left(\frac{y}{x}\right)^{1+\omega_0^B(x)} dx > F_2\left(v_2;\omega_1^B\right) = \int_0^{1}\int_0^{\min\{x,v_2\}} d\left(\frac{y}{x}\right)^{1+\omega_1^B(x)} dx$$

In the next examples Assumption 2 holds.

Example 2. *Each buyer independently draws k_B marginal values from an absolutely continuous, univariate, distribution $\widetilde{F}\left(v;\omega^B\right)$, $v \in [0,1]$, with the k-th marginal value being the k-th order statistic. Similarly, each seller independently draws k_S marginal costs from an absolutely continuous, univariate, distribution $\widetilde{G}\left(c;\omega^S\right)$, $c \in [0,1]$, with the k-th marginal cost being the (k_S+1-k)-th order statistic.*

We will only prove that (10) holds, as the proof that (11) is satisfied is similar. For buyers, the distributions of the k-th order statistic is

$$F_k(p;\omega^B) = k_B\binom{k_B-1}{k-1}\int_0^p \widetilde{f}\left(x;\omega^B\right)\left[1-\widetilde{F}\left(x;\omega^B\right)\right]^{k-1}\left(\widetilde{F}\left(x;\omega^B\right)\right)^{k_B-k} dx$$

Since $\{\boldsymbol{y}^B \in \mathcal{V} : y_1^B = \ldots = y_{k_B}^B = y \leqslant p\}$ is a subset of $\{\boldsymbol{y}^B \in \mathcal{V} : y_1^B \leqslant p\}$, we have:

$$\begin{aligned}\sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant p\}}\left|F(\boldsymbol{y}^B;\omega_0^B)-F(\boldsymbol{y}^B;\omega_1^B)\right| &\geqslant \sup_{\{y\in[0,1]:y\leqslant p\}}\left|F_1(y;\omega_0^B)-F_1(y;\omega_1^B)\right| \\ &= \sup_{\{y\in[0,1]:y\leqslant p\}}\left|\left(\widetilde{F}(y;\omega_0^B)\right)^{k_B}-\left(\widetilde{F}(y;\omega_1^B)\right)^{k_B}\right|\end{aligned}$$

Thus, to show that (10) holds it is sufficient to show that there exists $\kappa > 0$ such that:

$$\begin{aligned}\frac{1}{\kappa}\cdot k_B\binom{k_B-1}{k-1}\cdot\Bigg|&\int_0^p \widetilde{f}\left(x;\omega_0^B\right)\left[1-\widetilde{F}\left(x;\omega_0^B\right)\right]^{k-1}\left(\widetilde{F}\left(x;\omega_0^B\right)\right)^{k_B-k} dx \\ &-\int_0^p \widetilde{f}\left(x;\omega_1^B\right)\left[1-\widetilde{F}\left(x;\omega_1^B\right)\right]^{k-1}\left(\widetilde{F}\left(x;\omega_1^B\right)\right)^{k_B-k} dx\Bigg| \\ &\leqslant \sup_{\{y\in[0,1]:y\leqslant p\}}\left|\left(\widetilde{F}(y;\omega_0^B)\right)^{k_B}-\left(\widetilde{F}(y;\omega_1^B)\right)^{k_B}\right|\end{aligned}$$

First note that if the right hand side is positive, then the inequality holds for a sufficiently large κ. Second, if the right hand side is zero, then $\widetilde{F}(y;\omega_0^B) = \widetilde{F}(y;\omega_1^B)$ for all $y \leqslant p$ and the left hand side is also equal to zero, for all κ.

Example 3. *For all* $\omega^B \in \Omega^B$ *and all* $\omega^S \in \Omega^S$*, the multivariate distribution* F *associated with the measure* $\mu^{\omega^B} \in \mathcal{F}$ *and the multivariate reliability* R *associated with the measure* $\mu^{\omega^S} \in \mathcal{G}$ *are analytic functions.*

Consider the following open subsets of the domain of valuations and costs: $V(p) = \{\boldsymbol{v}^b \in \mathcal{V} : v_1^b < p\}$; $C(p) = \{\boldsymbol{c}^s \in \mathcal{C} : c_1^s > p\}$. In the limit, the DCA uncovers the value taken by the true distribution function F in the domain $V(p)$ for all $p \leqslant r^B$ and by the true reliability function R in the domain $C(p)$ for all $p \geqslant r^S$. By the identity theorem of analytic functions of multiple variables (e.g., see Krantz and Parks, 2002, p. 83), the true distribution and reliability functions and hence the values they take in their entire domain are also uncovered; if the two analytic functions agree (i.e., take the same value) on an open connected subset of their domain, then they are the same function (i.e., they take the same value at all points in their domain). It follows that when F and R are analytic Assumption 2 holds. As a corollary, it then also holds in the next example.

Example 4. *Each buyer type* $\boldsymbol{v}^b$ *is drawn from a distribution* $F\left(\boldsymbol{v}^b; \omega^B\right)$ *belonging to a parametric family of multivariate distributions, e.g., truncated normal, exponential, gamma, Dirichlet, Weibull, etc. Similarly, each seller type* $\boldsymbol{c}^s$ *is drawn from a distribution* $G\left(\boldsymbol{c}^s; \omega^S\right)$ *belonging to a parametric family of distributions.*

8.1 Convergence to Efficiency

Consider the estimation state at the beginning of round t in the discovery phase of the DCA. Let p^B be the buyers' clock price. We assume the designer computes $\omega_k^B(p^B)$ from (8) and uses the following as the estimated quantity demanded at price p:

$$\widehat{D}_t^n(p; p^B) = nN \sum_{k=1}^{k_B} \left[1 - F_k(p; \omega^B(p^B))\right] \tag{12}$$

Similarly, let p^S be the sellers' clock price; the designer computes $\omega^S(p^S)$ from (9) and uses the following as the estimated quantity supplied at price p:

$$\widehat{S}_t^n(p; p^S) = nM \sum_{k=1}^{k_S} \left[1 - R_k(p; \omega^S(p^S))\right] \tag{13}$$

Define the percentage efficiency loss of the DCA as 1 minus the ratio of total welfare in the DCA over maximum welfare (i.e., welfare under the efficient outcome). We show that the expected percentage efficiency loss converges to zero as $n \to \infty$. We also show that the designer's profit per-capita converges to zero; per-capita profit is the difference between total payments by buyers and total payments to sellers divided by the number of traders $n\,(N + M)$.

Theorem 3. *Under Assumptions 1 and 2, in the efficiency targeting version of the DCA, the expected percentage efficiency loss and the expected per-capita profit of the designer converge to zero as* $n \to \infty$.

Proof. See the appendix. □

8.2 Convergence to Maximum Profit Extraction

If the designer knew market demand and supply, he could choose a uniform price for buyers and a uniform price for sellers so as to maximize his profit. We will refer to the profit that would be generated by such a mechanism as the maximum profit.

Let $G_k(p;\omega^S) = 1 - R_k(p;\omega^S)$ be the distribution associated with reliability $R_k(\cdot)$. Assumption 3 is a strenghtening of Assumption 1.

Assumption 3. (*Monotonicity of Marginal Revenue and Procurement Cost*)
(a) *For all* $p \in (0,1)$, *all* $\epsilon > 0$, *all* $k \in \{1,...,k_B\} \cup \{1,...,k_S\}$, *all* $\omega^B \in \Omega^B$ *and all* $\omega^S \in \Omega^S$, *there exists* $\eta > 0$ *such that:*

$$f_k(p;\omega^B) = \lim_{\epsilon \to 0} \frac{F_k(p;\omega^B) - F_k(p-\epsilon;\omega^B)}{\epsilon} > \eta \tag{14}$$

$$g_k(p;\omega^S) = \lim_{\epsilon \to 0} \frac{G_k(p;\omega^S) - G_k(p-\epsilon;\omega^S)}{\epsilon} > \eta \tag{15}$$

(b) *For all* $\omega^B \in \Omega^B$ *and* $\omega^S \in \Omega^S$, $MR(p;\omega^B)$ *is a strictly increasing function of* p *and* $MC(p;\omega^S)$ *is a strictly increasing function of* p, *where:*

$$MR(p;\omega^B) = p + d\left(p;\omega^B\right) \cdot \frac{1}{\frac{\partial d(p;\omega^B)}{\partial p}} = p - \frac{\sum_{k=1}^{k_B}\left[1 - F_k(p;\omega^B)\right]}{\sum_{k=1}^{k_B} f_k(p;\omega^B)} \tag{16}$$

$$MC(p;\omega^S) = p + s\left(p;\omega^S\right) \cdot \frac{1}{\frac{\partial s(p;\omega^S)}{\partial p}} = p + \frac{\sum_{k=1}^{k_S} G_k(p;\omega^S)}{\sum_{k=1}^{k_S} g_k(p;\omega^S)} \tag{17}$$

Assumption 3(a) requires that the measures μ^{ω^B} and μ^{ω^S} are absolutely continuous with respect to Lebesgue measure and their Radon-Nikodym derivatives (densities) are bounded away from zero. Assumption 3(b) requires that the marginal revenue and marginal procurement cost of an increase in quantity traded are strictly monotone.

In the profit targeting DCA, to determine target prices the designer uses the estimated demand and supply in (12) and (13) together with the associated estimated marginal revenue and marginal procurement cost:

$$\widehat{MR}_t^n(p;p^B) = p - \frac{\sum_{k=1}^{k_B}\left[1 - F_k(p;\omega^B(p^B))\right]}{\sum_{k=1}^{k_B} f_k(p;\omega^B(p^B))} \tag{18}$$

$$\widehat{MC}_t^n(p;p^S) = p + \frac{\sum_{k=1}^{k_S} G_k(p;\omega^S(p^S))}{\sum_{k=1}^{k_S} g_k(p;\omega^B(p^S))} \tag{19}$$

In the profit targeting DCA, the stopping functions are $\sigma_t^B(\cdot) = \widehat{MR}_t^n(\cdot)$ and $\sigma_t^S(\cdot) = \widehat{MC}_t^n(\cdot)$.

The percentage loss in profit over maximum profit is defined as 1 minus the ratio of the designer's total profit over maximum profit.

Theorem 4. *Under Assumptions 2 and 3, in the profit targeting version of the DCA, the expected percentage loss in the designer profit over maximum profit converges to zero as* $n \to \infty$.

Proof. See the appendix. □

9 Conclusions

For the canonical economic model of a homogenous good market with multi-unit traders having multi-dimensional private information about their marginal values and costs, this paper proposes a solution to the problem faced by a market designer who wishes to target either an efficient allocation of resources without running a deficit or profit maximization. Our design consists of an estimation-based tâtonnement mechanism that makes bidding according to the true demand and supply schedules by all traders a dominant strategy equilibrium. Our double clock auction first determines reserve prices in the price discovery phase and then allocates the minimum of the quantity demanded and supplied at these reserve prices in two Ausubel auctions in the allocation phase. Under mild regularity conditions, the market outcome is asymptotically optimal; that is, it converges to either an efficient allocation of resources, or to the profit maximizing price-posting outcome, depending on the target pursued by the market maker.

The design we propose is flexible in important dimensions. While we focus on targeting either efficiency or profit, simple modifications of our double clock auction would permit the market maker to pursue an intermediate objective between profit and social surplus maximization. Our design also accommodates the incorporation of constraints on the aggregate quantities subsets of bidders may be allocated or may procure. In practice, quantity constraints may arise for a number of reasons, such as competitive concerns or technological constraints.

Progress in research is made one step at a time; in this paper, we have proposed an estimation-based market design for a homogeneous good market. Two avenues for future research using the approach of this paper seem particularly promising: expanding the setup to allow for heterogenous commodities as defined by Ausubel (2006) and studying versions of the assignment model of Shapley and Shubik (1972), which is simpler than what we studied here because agents trade at most one unit but challenging insofar as there is no natural ordering of agents according to their types.

Appendix

Proof of Theorem 3 (*Convergence to Efficiency*). Let ω_*^B and ω_*^S be the true parameters identifying the measures from which values and costs are drawn. Using the expectation operator $\mathbb{E}$ with respect to the true measures (with parameters ω_*^B and ω_*^S), we have: $\mathbb{E}\left[\mathbf{1}(v_k^b \leqslant p)\right] = F_k(p;\omega_*^B)$; $\mathbb{E}\left[\mathbf{1}(c_k^s \geqslant p)\right] = R_k(p;\omega_*^S)$; for $\boldsymbol{y}^B \in \mathcal{V}$, $\mathbb{E}\left[\mathbf{1}(\boldsymbol{v}^b \leqslant \boldsymbol{y}^B)\right] = F(\boldsymbol{y}^B;\omega_*^B)$; for $\boldsymbol{y}^S \in \mathcal{C}$, $\mathbb{E}\left[\mathbf{1}(\boldsymbol{c}^s \geqslant \boldsymbol{y}^S)\right] = R(\boldsymbol{y}^S;\omega_*^S)$. This observation allows us to state a simple consequence of Chebyshev's exponential inequality (see (42), page 69 of Shiryaev, 1996), which implies convergence of the empirical distributions and reliabilities to the true distributions and reliabilities as $n \to \infty$:

$$\Pr\left\{\left|F_k^n(p) - F_k(p;\omega_*^B)\right| \geqslant \epsilon\right\} \leqslant 2e^{-2\epsilon^2 nN} \tag{20}$$

$$\Pr\left\{\left|R_k^n(p) - R_k(p;\omega_*^S)\right| \geqslant \epsilon\right\} \leqslant 2e^{-2\epsilon^2 nM} \tag{21}$$

$$\Pr\left\{\left|F^n(\boldsymbol{y}^B;p) - F(\boldsymbol{y}^B;\omega_*^B)\right| \geqslant \epsilon\right\} \leqslant 2e^{-2\epsilon^2 nN} \tag{22}$$

$$\Pr\left\{\left|R^n(\boldsymbol{y}^S;p) - R(\boldsymbol{y}^S;\omega_*^S)\right| \geqslant \epsilon\right\} \leqslant 2e^{-2\epsilon^2 nM} \tag{23}$$

By setting $\epsilon = n^{\alpha-1/2}$ with $\alpha \in (0, 1/2)$, it is immediate to obtain the next lemma, which states that the empirical distributions and reliabilities converge to the true distributions and reliabilities at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$.

Lemma 1. *For all* $p \in (0,1), \boldsymbol{y}^B \in \mathcal{V}, \boldsymbol{y}^S \in \mathcal{C}$:

$$\Pr\left\{\left|F_k^n(p) - F_k(p;\omega_*^B)\right| < n^{\alpha-\frac{1}{2}}\right\} \to 1 \quad \textit{as} \quad n \to \infty \tag{24}$$

$$\Pr\left\{\left|R_k^n(p) - R_k(p;\omega_*^S)\right| < n^{\alpha-\frac{1}{2}}\right\} \to 1 \quad \textit{as} \quad n \to \infty \tag{25}$$

$$\Pr\left\{\left|F^n(\boldsymbol{y}^B;p) - F(\boldsymbol{y}^B;\omega_*^B)\right| < n^{\alpha-\frac{1}{2}}\right\} \to 1 \quad \textit{as} \quad n \to \infty \tag{26}$$

$$\Pr\left\{\left|R^n(\boldsymbol{y}^S;p) - R(\boldsymbol{y}^S;\omega_*^S)\right| < n^{\alpha-\frac{1}{2}}\right\} \to 1 \quad \textit{as} \quad n \to \infty \tag{27}$$

Define inverse market demand and supply as follows:

$$D^{n^{-1}}(q) = \{\max p : D^n(p) \geqslant q\} \tag{28}$$

$$S^{n^{-1}}(q) = \{\min p : S^n(p) \geqslant q\} \tag{29}$$

If buyers are on the short side (i.e., $D^n(r) = q$ where $r = r^B = r^S$ is the common reserve price), the total efficiency loss is bounded above by the following value:

$$\left|S^n(r) - q\right| \cdot \left|r - S^{n^{-1}}(q)\right| = \left|S^n(r) - D^n(r)\right| \cdot \left|r - S^{n^{-1}}\left(D^n(r)\right)\right| \tag{30}$$

If sellers are on the short side (i.e., $S^n(r) = q$), the efficiency loss is bounded above by the following value:

$$\left|D^n(r) - q\right| \cdot \left|D^{n^{-1}}(q) - r\right| = \left|S^n(r) - D^n(r)\right| \cdot \left|r - D^{n^{-1}}\left(S^n(r)\right)\right| \tag{31}$$

Consider the event in which buyers are on the short side of the market.

In the efficiency targeting DCA, the stopping functions $\sigma_t^B(\cdot)$ and $\sigma_t^S(\cdot)$ are the identity functions. Thus, the discovery phase of the DCA ends in an estimation state that follows either: (a) a double clock state when both clocks simultaneously stopped at the target price p^T that satisfies $\widehat{D}_t^n(p^T) = \widehat{S}_t^n(p^T)$ – in such a case the target price p^T becomes the buyers and sellers' reserve price in the second phase of the DCA, $r = r^B = r^S = p^T$; (b) a buyers' clock state that reached the target price $p^{BT} = p^S$, the current sellers' clock price – in such a case p^S becomes the common reserve price in the second phase of the DCA; or (c) a sellers' clock state that reached the target price $p^{ST} = p^B$, the current buyers' clock price – in such a case p^B becomes the reserve price in the second phase of the DCA.

When the discovery phase ends in case (a) it is $\widehat{D}^n(r) = \widehat{S}^n(r)$; in case (b) it is $\widehat{D}^n(r) \geqslant \widehat{S}^n(r)$; while in case (c) it is $\widehat{D}^n(r) \leqslant \widehat{S}^n(r)$. Let $\widehat{Z}^n(r) = \widehat{D}^n(r) - \widehat{S}^n(r)$. Then (30) reduces to:

$$\mathcal{L}^B = \Big|[S^n(r) - \widehat{S}^n(r)] - [D^n(r) - \widehat{D}^n(r)] - \widehat{Z}^n(r)\Big| \cdot \Big|r - S^{n^{-1}}\big(D^n(r)\big)\Big| \tag{32}$$

If sellers are on the short side (i.e., $S^n(r) = q$), then (31) reduces to:

$$\mathcal{L}^S = \Big|[S^n(r) - \widehat{S}^n(r)] - [D^n(r) - \widehat{D}^n(r)] - \widehat{Z}^n(r)\Big| \cdot \Big|r - D^{n^{-1}}\big(S^n(r)\big)\Big| \tag{33}$$

Since convergence follows from the same arguments when buyers or sellers are on the short side, we will only present the proof for the case in which buyers are on the short side. By (32) we have $\mathcal{L}^B = \Delta_1^B \cdot \Delta_2^B$ where:

$$\Delta_1^B = \Big|[S^n(r) - \widehat{S}^n(r)] - [D^n(r) - \widehat{D}^n(r)] - \widehat{Z}^n(r)\Big|$$
$$\Delta_2^B = \Big|r - S^{n^{-1}}\big(D^n(r)\big)\Big|$$

Since total welfare under the efficient, Walrasian, allocation grows at rate n, we need to prove that $\mathcal{L}^B/n$ converges to zero as $n \to \infty$. To do so, we demonstrate first that $\Delta_1^B/n \to 0$ and then that $\Delta_2^B \to 0$ as $n \to \infty$ (both at rate $n^{\alpha-\frac{1}{2}}$).

Lemma 2. $\Pr\left\{\frac{\Delta_1^B}{n} < n^{\alpha-\frac{1}{2}}\right\} \to 1$ *as* $n \to \infty$.

Proof. Note that:

$$\frac{\Delta_1^B}{n} \leqslant \left(\frac{1}{n}\Big|S^n(r) - \widehat{S}^n(r)\Big| + \frac{1}{n}\Big|D^n(r) - \widehat{D}^n(r)\Big| + \left|\frac{\widehat{Z}^n(r)}{n}\right|\right)$$

To prove Lemma 2 we proceed in two steps. First we show that the first two terms in brackets converge to zero at rate $n^{\alpha-\frac{1}{2}}$, then we show that $\left|\frac{\widehat{Z}^n(r)}{n}\right|$ also converges to zero at rate $n^{\alpha-\frac{1}{2}}$.

Step 1. Let ω_*^B and ω_*^S be the "true" parameters indicating the measures from which buyers and sellers values are drawn. Observe that:

$$\frac{1}{n}\left|D^n(r)-\widehat{D}^n(r)\right| = N\left|\sum_{k=1}^{k^B}\left[F_k^n(r)-F_k(r;\omega^B(r))\right]\right|$$
$$\frac{1}{n}\left|S^n(r)-\widehat{S}^n(r)\right| = M\left|\sum_{k=1}^{k^S}\left[R_k^n(r)-R_k(r;\omega^S(r))\right]\right|$$

We will show that the expected value of $\left|\sum_{k=1}^{k^B}\left[F_k^n(r)-F_k(r;\omega^B(r))\right]\right| \to 0$ at rate $n^{\alpha-\frac{1}{2}}$ as $n\to\infty$; the proof that $\left|\sum_{k=1}^{k^S}\left[R_k^n(r)-R_k(r;\omega^S(r))\right]\right| \to 0$ at the same rate is analogous and we omit it. We have:

$$\begin{aligned}\left|\sum_{k=1}^{k^B}\left[F_k^n(r)-F_k(r;\omega^B(r))\right]\right| &\leqslant \sum_{k=1}^{k^B}\left|F_k^n(r)-F_k(r;\omega^B(r))\right|\\ &= \sum_{k=1}^{k^B}\left|F_k^n(r)-F_k(r;\omega_*^B)+F_k(r;\omega_*^B)-F_k(r;\omega^B(r))\right|\\ &\leqslant \sum_{k=1}^{k^B}\left(\left|F_k^n(r)-F_k(r;\omega_*^B)\right|+\left|F_k(r;\omega_*^B)-F_k(r;\omega^B(r))\right|\right)\end{aligned}$$

By (24), the expected value of $\left|F_k^n(r)-F_k(r;\omega_*^B)\right| \to 0$ at rate $n^{\alpha-\frac{1}{2}}$ as $n\to\infty$. Thus, it only remains to consider the second term on the rhs of the last inequality:

$$\begin{aligned}&\left|F_k(r;\omega_*^B)-F_k(r;\omega^B(r))\right|\\ &\quad\leqslant \kappa\cdot\sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant r\}}\left|F(\boldsymbol{y}^B;\omega_*^B)-F(\boldsymbol{y}^B;\omega^B(r))\right| \qquad (34)\\ &\quad= \kappa\cdot\sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant r\}}\left|F(\boldsymbol{y}^B;\omega_*^B)-F^n(\boldsymbol{y}^B,r)+F^n(\boldsymbol{y}^B,r)-F(\boldsymbol{y}^B;\omega^B(r))\right|\\ &\quad\leqslant \kappa\cdot\left(\sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant r\}}\left|F(\boldsymbol{y}^B;\omega_*^B)-F^n(\boldsymbol{y}^B,r)\right|+\sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1\leqslant r\}}\left|F^n(\boldsymbol{y}^B,r)-F(\boldsymbol{y}^B;\omega^B(r))\right|\right)\end{aligned}$$

where (34) follows from (10). The expected value of $\sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant r\}}\left|F(\boldsymbol{y}^B;\omega_*^B)-F^n(\boldsymbol{y}^B,r)\right|$ $\to 0$ at rate $n^{\alpha-\frac{1}{2}}$ as $n\to\infty$, by (26) in Lemma 1. The remaining term to consider is:

$$\begin{aligned}\sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1\leqslant r\}}\left|F^n(\boldsymbol{y}^B,r)-F(\boldsymbol{y}^B;\omega^B(r))\right| &= \min_{\omega^B\in\Omega^B}\sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant r\}}\left|F^n(\boldsymbol{y}^B,r)-F(\boldsymbol{y}^B;\omega^B)\right| \qquad (35)\\ &\leqslant \sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant r\}}\left|F^n(\boldsymbol{y}^B,r)-F(\boldsymbol{y}^B;\omega_*^B)\right|\end{aligned}$$

where (35) follows from (8). This completes the first step in the proof of Lemma 2, as the expected value of the term on the rhs of the inequality converges to zero at rate $n^{\alpha-\frac{1}{2}}$ as $n\to\infty$ by (26).

Step 2: We now show that $\left|\widehat{Z}^n(r)/n\right| \to 0$ at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$. We only consider the case $\widehat{S}^n(r) < \widehat{D}^n(r)$, as the proof for the case $\widehat{S}^n(r) > \widehat{D}^n(r)$ is analogous. Note that $\widehat{S}^n(r) < \widehat{D}^n(r)$ implies $r = p^S$ and that the estimate $\widehat{S}^n(r)$ is obtained after the last seller drops out at r in the discovery phase. Define $\widehat{S}^n_-(r)$ as the supply estimate in the round before the last seller drops out. Define the estimators associated with $\widehat{S}^n(r)$ and $\widehat{S}^n_-(r)$:

$$\omega^S(r) \in \arg\min_{\omega^S \in \Omega^S} \sup_{\{\boldsymbol{y}^S \in \mathcal{C}: y_1^S \geqslant r\}} \left|R^n(\boldsymbol{y}^S; r) - R(\boldsymbol{y}^S; \omega^S)\right|$$

$$\omega^S_-(r) \in \arg\min_{\omega^S \in \Omega^S} \sup_{\{\boldsymbol{y}^S \in \mathcal{C}: y_1^S \geqslant r\}} \left|R^n(\boldsymbol{y}^S; r) - \frac{1}{nM} - R(\boldsymbol{y}^S; \omega^S)\right|$$

Since the buyers' clock stops at price $r = p^S$ without any additional buyer dropping out, $\widehat{D}^n(r)$ does not change while the buyers' clock price decreases and the sellers' clock price is set at r. Thus, it must be $\widehat{S}^n_-(r) \geqslant \widehat{D}^n(r)$, with a strict inequality if the seller's clock is the only clock moving in the round when the last seller drops out. It follows that there exists $\lambda(r) \in [0,1]$ such that $[1-\lambda(r)]\widehat{S}^n(r) + \lambda(r)\widehat{S}^n_-(r) = \widehat{D}^n(r)$. Hence:

$$\begin{aligned}
\left|\frac{\widehat{Z}^n(r)}{n}\right| &= \frac{1}{n}\left|[1-\lambda(r)]\widehat{S}^n(r) + \lambda(r)\widehat{S}^n_-(r) - \widehat{S}^n(r)\right| \\
&= \frac{\lambda(r)}{n}\left|\widehat{S}^n_-(r) - \widehat{S}^n(r)\right| \\
&= \frac{\lambda(r)}{n} nM \left|\sum_{k=1}^{k^B}\left[R_k\left(r; \omega^S_-(r)\right) - R_k\left(r; \omega^S(r)\right)\right]\right| \\
&\leqslant \lambda(r) M \sum_{k=1}^{k^B}\left|R_k\left(r; \omega^S_-(r)\right) - R_k\left(r; \omega^S(r)\right)\right|
\end{aligned}$$

Consider an arbitrary term on the rhs of the inequality. By (11) we have:

$$\begin{aligned}
&\left|R_k\left(r; \omega^S_-(r)\right) - R_k\left(r; \omega^S(r)\right)\right| \\
&\quad\leqslant \kappa \cdot \sup_{\{\boldsymbol{y}^S \in \mathcal{C}: y_1^S \geqslant r\}} \left|R\left(\boldsymbol{y}^S; \omega^S_-(r)\right) - R\left(\boldsymbol{y}^S; \omega^S(r)\right)\right| \\
&\quad\leqslant \kappa \cdot \left(\sup_{\{\boldsymbol{y}^S \in \mathcal{C}: y_1^S \geqslant r\}} \left|R\left(\boldsymbol{y}^S; \omega^S_-(r)\right) - R^n\left(\boldsymbol{y}^S; r\right)\right| + \sup_{\{\boldsymbol{y}^S \in \mathcal{C}: y_1^S \geqslant r\}} \left|R^n\left(\boldsymbol{y}^S; r\right) - R\left(\boldsymbol{y}^S; \omega^S(r)\right)\right|\right)
\end{aligned} \tag{36}$$

Consider the second term on the rhs of (36); recalling that ω^S_* is the true parameter:

$$\begin{aligned}
\sup_{\{\boldsymbol{y}^S \in \mathcal{C}: y_1^S \geqslant r\}} \left|R\left(\boldsymbol{y}^S; \omega^S(r)\right) - R^n\left(\boldsymbol{y}^S; r\right)\right| &= \min_{\omega^S \in \Omega^S} \sup_{\{\boldsymbol{y}^S \in \mathcal{C}: y_1^S \geqslant r\}} \left|R\left(\boldsymbol{y}^S; \omega^S\right) - R^n\left(\boldsymbol{y}^S; r\right)\right| \\
&\leqslant \sup_{\{\boldsymbol{y}^S \in \mathcal{C}: y_1^S \geqslant r\}} \left|R\left(\boldsymbol{y}^S; \omega^S_*\right) - R^n\left(\boldsymbol{y}^S; r\right)\right|
\end{aligned}$$

By (27), the expected value of the rhs of the last inequality converges to zero at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$. Now consider the first term on the rhs of (36):

$$\begin{aligned}
&\sup_{\{\boldsymbol{y}\in\mathcal{C}:y_1^S\geqslant r\}} \left|R\left(\boldsymbol{y}^S;\omega_-^S(r)\right) - R^n\left(\boldsymbol{y}^S;r\right)\right| \\
&\qquad = \sup_{\{\boldsymbol{y}^S\in\mathcal{C}:y_1^S\geqslant r\}} \left|R\left(\boldsymbol{y}^S;\omega_-^S(r)\right) - \frac{1}{nM} + \frac{1}{nM} - R^n\left(\boldsymbol{y}^S;r\right)\right| \\
&\qquad \leqslant \sup_{\{\boldsymbol{y}^S\in\mathcal{C}:y_1^S\geqslant r\}} \left|R\left(\boldsymbol{y}^S;\omega_-^S(r)\right) - \frac{1}{nM} - R^n\left(\boldsymbol{y}^S;r\right)\right| + \frac{1}{nM} \\
&\qquad = \min_{\omega^S\in\Omega^S} \sup_{\{\boldsymbol{y}^S\in\mathcal{C}:y_1^S\geqslant r\}} \left|R\left(\boldsymbol{y}^S;\omega^S\right) - \frac{1}{nM} - R^n\left(\boldsymbol{y}^S;r\right)\right| + \frac{1}{nM} \\
&\qquad \leqslant \sup_{\{\boldsymbol{y}^S\in\mathcal{C}:y_1^S\geqslant r\}} \left|R\left(\boldsymbol{y}^S;\omega_*^S\right) - \frac{1}{nM} - R^n\left(\boldsymbol{y}^S;r\right)\right| + \frac{1}{nM} \\
&\qquad \leqslant \sup_{\{\boldsymbol{y}^S\in\mathcal{C}:y_1^S\geqslant r\}} \left|R\left(\boldsymbol{y}^S;\omega_*^S\right) - R^n\left(\boldsymbol{y}^S;r\right)\right| + \frac{2}{nM}
\end{aligned}$$

Again, by (27) the expected value of the rhs converges to zero at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$. This concludes the proof of the second and last step in the proof of Lemma 2. □

To finish the proof of Theorem 3, it remains to prove the following lemma.

Lemma 3. $\Pr\left\{\Delta_2^B < n^{\alpha-\frac{1}{2}}\right\} \to 1$ *as* $n \to \infty$.

Proof. Since we are considering the case in which buyers are on the short side of the market, it is $r \geqslant S^{n^{-1}}(D^n(r))$; hence we have:

$$\begin{aligned}
\Pr&\left\{\Delta_2^B = \left|r - S^{n^{-1}}\left(D^n(r)\right)\right| < \epsilon\right\} = \Pr\left\{r - \epsilon < S^{n^{-1}}\left(D^n(r)\right)\right\} \\
&\geqslant \Pr\left\{S^n(r-\epsilon) < D^n(r)\right\} \\
&= \Pr\Big\{S\left(r;\omega_*^S\right) - S\left(r-\epsilon;\omega_*^S\right) > \\
&\qquad + \Big(S\left(r;\omega_*^S\right) - S^n(r)\Big) - \Big(S\left(r-\epsilon;\omega_*^S\right) - S^n(r-\epsilon)\Big) + \Big(S^n(r) - D^n(r)\Big)\Big\} \\
&= \Pr\Big\{\sum_{k=1}^{k^S}\left[R_k(r-\epsilon;\omega_*^S) - R_k(r;\omega_*^S)\right] > \\
&\qquad + \sum_{k=1}^{k^S}\left[R_k^n(r) - R_k(r;\omega_*^S)\right] + \sum_{k=1}^{k^S}\left[R_k(r-\epsilon;\omega_*^S) - R_k^n(r-\epsilon)\right] + \frac{\Delta_1^B}{n}\Big\} \\
&\geqslant \Pr\left\{\sum_{k=1}^{k^S}\left|R_k^n(r) - R_k(r;\omega_*^S)\right| + \sum_{k=1}^{k^S}\left|R_k(r-\epsilon;\omega_*^S) - R_k^n(r-\epsilon)\right| + \frac{\Delta_1^B}{n} < \eta\epsilon\right\} \qquad (37)
\end{aligned}$$

where: the first inequality holds because, by (29), $S^n\left(S^{n^{-1}}\left(D^n(r)\right)\right) \geqslant D^n(r)$; the second equality follows because we are considering the case in which buyers are on the short side and hence $S^n(r) \geqslant D^n(r)$; the last inequality holds by (5) in Assumption 1. By setting $\epsilon = \frac{1}{\eta} n^{\alpha-\frac{1}{2}}$, this concludes the proof of Lemma 3, because by (25) in Lemma 1 and Lemma 2 the terms on the lhs of the expression inside the curly brackets of (37) converge to zero at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$. □

By setting $\gamma = 2\alpha$, we obtain that the the percentage efficiency loss $\mathcal{L}^B/n$ converges to zero at rate $n^{\gamma-1}$ for any γ arbitrarily close to zero. In addition, the per-capita profit of the designer, which equals Δ_2^B, converges to zero at rate $n^{\alpha-\frac{1}{2}}$. This concludes the proof of Theorem 3. □

Proof of Theorem 4 (*Convergence to Maximum Profit Extraction*). At reserve prices r^B, r^S, the designer's profit is:

$$\Pi^n\left(r^B, r^S\right) = \left(r^B - r^S\right) \cdot \min\left\{D^n\left(r^B\right), S^n\left(r^S\right)\right\} \tag{38}$$

Maximum profit is:

$$\Pi^n\left(r^B_M, r^S_M\right) = \max_{r^B, r^S \in (0,1)} \Pi^n\left(r^B, r^S\right) \tag{39}$$

Note that it must be $D^n(r^B_M) = S^n(r^S_M)$, otherwise profit could be increased by either raising r^B_M or lowering r^S_M. Let ω^B_* and ω^S_* be the true parameters of the buyers and sellers' measures. By Lemma 1, $D^n(r^B_M)/nN$ converges to $d(r^B_M; \omega^B_*)$ and $S^n(r^S_M)/nM$ converges to $s(r^S_M; \omega^S_*)$ at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$. Hence maximum profit converges at rate $n^{\alpha-\frac{1}{2}}$ to:

$$\Pi^n_* = nN\left(r^B_* - r^S_*\right) d\left(r^B_*; \omega^B_*\right) = nM\left(r^B_* - r^S_*\right) s\left(r^S_*; \omega^S_*\right) \tag{40}$$

where r^B_* and r^S_* are the reserve prices that would maximize profit if true per-capita demand and supply where $d(p; \omega^B_*)$ and $s(p; \omega^S_*)$; that is, they are the reserve prices that solve the following system of equations, which by Assumption 3 are unique:

$$\left\{\begin{array}{rcl} MR(r^B_*; \omega^B_*) & = & MC(r^S_*; \omega^S_*) \\ nNd(r^B_*; \omega^B_*) & = & nMs(r^S_*; \omega^S_*) \end{array}\right. \tag{41}$$

We need to show that $\frac{\Pi^n_* - \Pi^n\left(r^B, r^S\right)}{\Pi^n_*} \to 0$ as $n \to \infty$.

In the profit targeting DCA, the stopping functions are $\sigma^B_t(\cdot) = \widehat{MR}^n_t(\cdot)$ and $\sigma^S_t(\cdot) = \widehat{MC}^n_t(\cdot)$. The discovery phase may end in four different ways. In the first three cases the reserve prices r^B and r^S satisfy $\widehat{MR}^n(r^B) = \widehat{MC}^n(r^S)$. In case (a) it is $\widehat{D}^n(r^B) = \widehat{S}^n(r^S)$; the discovery phase ends after a double clock state with both clocks simultaneously stopping at the target prices p^{BT} and p^{ST} which become the reserve prices, $r^B = p^{BT}$ and $r^S = p^{ST}$. In case (b) it is $\widehat{D}^n(r^B) \geqslant \widehat{S}^n(r^S)$; the discovery phase ends after a buyers' clock state with the clock stopping at the target price p^{BT} which becomes the buyers' reserve price, $r^B = p^{BT}$, while the current sellers' clock price becomes the sellers' reserve price $r^S = p^S$. In case (c) it is $\widehat{D}^n(r^B) \leqslant \widehat{S}^n(r^S)$; the discovery phase ends after a sellers' clock state with the clock stopping at the target price p^{ST} which becomes the sellers' reserve price, $r^S = p^{ST}$, while the current buyers' clock price becomes the buyers' reserve price $r^B = p^B$. In the fourth case, case (d), the discovery phase ends after a round when a trader drops out and demand or supply re-estimation yields $\widehat{MR}_t^n(p^B) > \widehat{MC}_t^n(p^S)$ where p^B and p^S are the current clock prices, which become the reserve prices, $r^B = p^B, r^S = p^S$.

Letting $\widehat{Z}^n\left(r^B, r^S\right) = \widehat{D}^n(r^B) - \widehat{S}^n(r^S)$ and $\widehat{Y}^n\left(r^B, r^S\right) = \widehat{MR}^n(r^B) - \widehat{MC}^n(r^S)$, by definition we have:

$$\begin{cases} MR(r^B;\omega^B(r^B)) &= MC(r^S;\omega^S(r^S)) + \widehat{Y}^n\left(r^B, r^S\right) \\ nNd(r^B;\omega^B(r^B)) &= nMs(r^S;\omega^S(r^S)) + \widehat{Z}^n\left(r^B, r^S\right) \end{cases} \tag{42}$$

and (41) and (42) can be rewritten as follows:

$$\begin{cases} r_*^B - \frac{\sum_1^{k^B}\left[1-F_k\left(r_*^B;\omega_*^B\right)\right]}{\sum_1^{k^B} f_k\left(r_*^B;\omega_*^B\right)} &= r_*^S + \frac{\sum_1^{k^S} G_k\left(r_*^S;\omega_*^S\right)}{\sum_1^{k^S} g_k\left(r_*^S;\omega_*^S\right)} \\ N\sum_1^{k^B}\left[1-F_k\left(r_*^B;\omega_*^B\right)\right] &= M\sum_1^{k^S} G_k\left(r_*^S;\omega_*^S\right) \end{cases} \tag{43}$$

$$\begin{cases} r^B - \frac{\sum_1^{k^B}\left[1-F_k\left(r^B;\omega^B(r^B)\right)\right]}{\sum_1^{k^B} f_k(r^B;\omega^B(r^B))} &= r^S + \frac{\sum_1^{k^S} G_k\left(r^S;\omega^S(r^S)\right)}{\sum_1^{k^S} g_k(r^S;\omega^S(r^S))} + \widehat{Y}^n\left(r^B, r^S\right) \\ N\sum_1^{k^B}\left[1-F_k\left(r^B;\omega^B(r^B)\right)\right] &= M\sum_1^{k^S} G_k\left(r^S;\omega^S(r^S)\right) + \frac{\widehat{Z}^n\left(r^B, r^S\right)}{n} \end{cases} \tag{44}$$

We begin with a lemma and then make two observations.

Lemma 4. *$\widehat{Y}^n(r^B, r^S)$ converges to zero at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$.*

Proof. Recall that $\widehat{Y}^n\left(r^B, r^S\right) \neq 0$ only when the discovery phase ends under case (d) with $\widehat{Y}^n\left(r^B, r^S\right) > 0$. In such a case, the estimate of either marginal revenue or marginal cost must be obtained after either a buyer drops out at r^B, or a seller drops out at r^S. As the proof of the two cases is analogous, we will only present the case when it is a buyer dropping out at r^B that pushes estimated marginal revenue above estimated marginal cost. Let $\widehat{MR}_-^n(r^B)$ be the marginal cost estimate before the last buyer drops out. The estimators associated with $\widehat{MR}^n(r^B)$ and $\widehat{MR}_-^n(r^B)$ are:

$$\omega^B(r^B) \in \arg\min_{\omega^B\in\Omega^B} \sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant r^B\}} \left|F^n(\boldsymbol{y}^B;r^B) - F(\boldsymbol{y}^B;\omega^B)\right|$$

$$\omega_-^B(r^B) \in \arg\min_{\omega^B\in\Omega^B} \sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant r^B\}} \left|F^n(\boldsymbol{y}^B;r^B) - \frac{1}{nN} - F(\boldsymbol{y}^B;\omega^B)\right|$$

Just before the last buyer drops out at r^B it must be $\widehat{MR}^n_-(r^B) \leqslant \widehat{MC}^n(r^S)$. Thus, there exists $\lambda\left(r^B, r^S\right) \in [0,1]$ such that $\left(1-\lambda\left(r^B, r^S\right)\right)\widehat{MR}^n\left(r^B\right) + \lambda\left(r^B, r^S\right)\widehat{MR}^n_-\left(r^B\right) = \widehat{MC}^n\left(r^S\right)$, and:

$$\begin{aligned}
\widehat{Y}^n\left(r^B, r^S\right) &= \widehat{MR}^n\left(r^B\right) - \left(1-\lambda\left(r^B, r^S\right)\right)\widehat{MR}^n\left(r^B\right) - \lambda\left(r^B, r^S\right)\widehat{MR}^n_-\left(r^B\right) \\
&= \lambda\left(r^B, r^S\right))\left(\widehat{MR}^n\left(r^B\right) - \widehat{MR}^n_-\left(r^B\right)\right) \\
&= \lambda\left(r^B, r^S\right)\left(\frac{\sum_{k=1}^{k^B}\left[1 - F_k\left(r^B; \omega^B_-\left(r^B\right)\right)\right]}{\sum_{k=1}^{k^B} f_k\left(r^B; \omega^B_-\left(r^B\right)\right)} - \frac{\sum_{k=1}^{k^B}\left[1 - F_k\left(r^B; \omega^B\left(r^B\right)\right)\right]}{\sum_{k=1}^{k^B} f_k\left(r^B; \omega^B\left(r^B\right)\right)}\right)
\end{aligned}$$

To conclude the proof of the lemma we will show that, for each k, $F_k\left(r^B; \omega^B_-\left(r^B\right)\right)$ converges to $F_k\left(r^B; \omega^B\left(r^B\right)\right)$ and $f_k\left(r^B; \omega^B_-\left(r^B\right)\right)$ converges to $f_k\left(r^B; \omega^B\left(r^B\right)\right)$ at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$. As the proofs of these two claims are analogous, we will only present the proof of the latter. Note that for δ sufficiently small $f_k\left(r^B; \omega^B(r^B)\right)$ can be approximated by $\frac{F_k\left(r^B+\delta;\omega^B(r^B)\right) - F_k\left(r^B-\delta;\omega^B(r^B)\right)}{2\delta}$; in particular, we can choose δ to be a function that converges to zero at rate $n^{-\alpha}$, guaranteeing convergence to the density as n grows. By (10) we have:

$$\begin{aligned}
&\left|\frac{F_k\left(r^B+\delta;\omega^B_-(r^B)\right) - F_k\left(r^B-\delta;\omega^B_-(r^B)\right)}{2\delta} - \frac{F_k\left(r^B+\delta;\omega^B(r^B)\right) - F_k\left(r^B-\delta;\omega^B(r^B)\right)}{2\delta}\right| \\
=&\left|\frac{F_k\left(r^B+\delta;\omega^B_-(r^B)\right) - F_k\left(r^B+\delta;\omega^B(r^B)\right)}{2\delta} + \frac{F_k\left(r^B-\delta;\omega^B(r^B)\right) - F_k\left(r^B-\delta;\omega^B_-(r^B)\right)}{2\delta}\right| \\
\leqslant& \frac{\kappa}{2\delta}\left(\sup_{\{\boldsymbol{y}^B\in\mathcal{V}: y_1^B \leqslant r^B+\delta\}}\left|F\left(\boldsymbol{y}^B; \omega^B_-\left(r^B\right)\right) - F\left(\boldsymbol{y}^B; \omega^B\left(r^B\right)\right)\right|\right. \\
&\left. + \sup_{\{\boldsymbol{y}^B\in\mathcal{V}: y_1^B \leqslant r^B-\delta\}}\left|F\left(\boldsymbol{y}^B; \omega^B_-\left(r^B\right)\right) - F\left(\boldsymbol{y}^B; \omega^B\left(r^B\right)\right)\right|\right) \qquad (45)
\end{aligned}$$

We claim that both terms in the brackets on the rhs of (45) converge to zero at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$, thus guaranteeing convergence of the rhs of (45) to zero, as δ goes to zero at rate $n^{-\alpha}$. Since the proofs are identical, we will only present the proof for the first term. We have:

$$\begin{aligned}
&\sup_{\{\boldsymbol{y}^B\in\mathcal{V}: y_1^B \leqslant r^B+\delta\}}\left|F\left(\boldsymbol{y}^B; \omega^B_-\left(r^B\right)\right) - F\left(\boldsymbol{y}^B; \omega^B\left(r^B\right)\right)\right| \\
\leqslant& \left(\sup_{\{\boldsymbol{y}^B\in\mathcal{V}: y_1^B \leqslant r^B+\delta\}}\left|F\left(\boldsymbol{y}^B; \omega^B_-\left(r^B\right)\right) - F^n\left(\boldsymbol{y}^B; r^B\right)\right| + \sup_{\{\boldsymbol{y}^B\in\mathcal{V}: y_1^B \leqslant r^B+\delta\}}\left|F^n\left(\boldsymbol{y}^B; r^B\right) - F\left(\boldsymbol{y}^B; \omega^B\left(r^B\right)\right)\right|\right) \qquad (46)
\end{aligned}$$

Recalling that ω^S_* is the true parameter, the second term on the rhs of (46) is:

$$\begin{aligned}
\sup_{\{\boldsymbol{y}^B\in\mathcal{V}: y_1^B \leqslant r^B+\delta\}}\left|F\left(\boldsymbol{y}^B; \omega^B\left(r^B\right)\right) - F^n\left(\boldsymbol{y}^B; r^B\right)\right| &= \min_{\omega^B\in\Omega^B}\sup_{\{\boldsymbol{y}^B\in\mathcal{V}: y_1^B \leqslant r^B+\delta\}}\left|F\left(\boldsymbol{y}^B; \omega^B\right) - F^n\left(\boldsymbol{y}^B; r^B\right)\right| \\
&\leqslant \sup_{\{\boldsymbol{y}^B\in\mathcal{V}: y_1^B \leqslant r^B+\delta\}}\left|F\left(\boldsymbol{y}^B; \omega^B_*\right) - F^n\left(\boldsymbol{y}^B; r^B\right)\right|
\end{aligned}$$

By (26) in Lemma 1, the expected value of the rhs of the last inequality converges to zero at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$. Now consider the first term on the rhs of (46):

$$\sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant r^B+\delta\}} \left|F\left(\boldsymbol{y}^B;\omega_-^B(r)\right) - F^n\left(\boldsymbol{y}^B;r^B\right)\right|$$

$$\leqslant \sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant r^B+\delta\}} \left|F\left(\boldsymbol{y}^B;\omega_-^B\left(r^B\right)\right) - \frac{1}{nN} - F^n\left(\boldsymbol{y}^B;r^B\right)\right| + \frac{1}{nN}$$

$$= \min_{\omega^B\in\Omega^B} \sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant r^B+\delta\}} \left|F\left(\boldsymbol{y}^B;\omega^B\right) - \frac{1}{nN} - F^n\left(\boldsymbol{y}^B;r^B\right)\right| + \frac{1}{nN}$$

$$\leqslant \sup_{\{\boldsymbol{y}^B\in\mathcal{V}:y_1^B\leqslant r^B+\delta\}} \left|F\left(\boldsymbol{y}^B;\omega_*^B\right) - F^n\left(\boldsymbol{y}^B;r^B\right)\right| + \frac{2}{nN}$$

Again, by (26) the expected value of the rhs converges to zero at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$. This concludes the proof of Lemma 4. □

Observation 1: The argument developed in step 1 of Lemma 2 with (34) and (35) shows that $F_k\left(r^B;\omega^B(r^B)\right)$ converges to $F_k\left(r^B;\omega_*^B\right)$ at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$; an analogous argument shows that $G_k\left(r^S;\omega^S(r^S)\right)$ converges to $G_k\left(r^S;\omega_*^S\right)$ at the same rate. As a consequence, convergence at rate $n^{\alpha-\frac{1}{2}}$ also obtains for the densities; that is, $f_k\left(r^B;\omega^B(r^B)\right)$ converges to $f_k\left(r^B;\omega_*^B\right)$ and $f_k\left(r^B;\omega^B(r^B)\right)$ converges to $g_k\left(r^S;\omega_*^S\right)$.[50]

Observation 2: $\widehat{Z}^n\left(r^B,r^S\right)/n$ converges to zero at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$. The proof is essentially identical to the proof of step 2 in Lemma 2 that $\left|\widehat{Z}^n(r)/n\right|$ converges to zero, with the only difference that the reserve prices for buyers and sellers, r^B and r^S, are different rather than being both equal to r.

Let $O_1\left(n^{\alpha-\frac{1}{2}}\right)$ and $O_2\left(n^{\alpha-\frac{1}{2}}\right)$ be two functions that converge to zero at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$. As a consequence of the two observations and Lemma 4, (44) can be rewritten as:

$$\begin{cases} \left(r^B - \frac{\sum_1^{k^B}\left[1-F_k\left(r^B;\omega_*^B\right)\right]}{\sum_1^{k^B} f_k\left(r^B;\omega_*^B\right)}\right) - \left(r^S + \frac{\sum_1^{k^S} G_k\left(r^S;\omega_*^S\right)}{\sum_1^{k^S} g_k\left(r^S;\omega_*^S\right)}\right) = O_1\left(n^{\alpha-\frac{1}{2}}\right) \\ \left(N\sum_1^{k^B}\left[1-F_k\left(r^B;\omega_*^B\right)\right]\right) - \left(M\sum_1^{k^S} G_k\left(r^S;\omega_*^S\right)\right) = O_2\left(n^{\alpha-\frac{1}{2}}\right) \end{cases} \tag{47}$$

By Assumption 3, the lhs of the first equation is monotone increasing in r^B and decreasing in r^S, while the lhs of the second equation is monotone decreasing in both r^B and r^S. We conclude that (47) converges to (43) and that its solution (r^B,r^S) converges to (r_*^B,r_*^S) at rate $n^{\alpha-\frac{1}{2}}$ as $n \to \infty$. As a result, the designer's expected percentage loss in profit over maximum profit, $\frac{\Pi_*^n-\Pi^n\left(r^B,r^S\right)}{\Pi_*^n}$ converges to zero at rate $n^{2\alpha-1}$ for any α arbitrarily close to zero. This concludes the proof of Theorem 4. □

[50]To see this we can use an argument like the one developed in the proof of Lemma 4; $f_k\left(r^B;\omega^B(r^B)\right)$ can be approximated by $\frac{F_k\left(r^B+\delta;\omega^B(r^B)\right)-F_k\left(r^B-\delta;\omega^B(r^B)\right)}{2\delta}$ with δ sufficiently small, and a similar approximation holds for $g_k\left(r^S;\omega^S(r^S)\right)$.

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